Gravitational theories and cosmology

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This came from the introduction chapter to my PhD thesis: "*Generalized perturbations in modified gravity and dark energy.* This is supposed to be relatively self-contained: please get in touch if you think something should be added or clarified.

Contents

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I. INTRODUCTION

Gravity is one of the most familiar of the fundamental forces of nature. It is the presence of gravity which enables the earth to orbit the Sun and keeps objects firmly planted on the surface of the earth. Whilst gravity is the weakest of the fundamental forces, it is gravity which dictates the structure and shape of the universe on a vast range of scales.

For a long time, the vast expanses of the natural world was imagined as a spacetime arena, upon which objects with mass were placed and moved; spacetime did not respond to the presence of the massive objects. The motion of a particle in spacetime was affected by its vicinity to massive objects which exerted gravitational forces upon the particle. On the whole, this paradigm was very successful and conceptually very simple, but was found to be flawed in situations where the gravitational forces were very strong. This picture was supplanted by Einstein's theory of gravity, where the important conceptual difference is that spacetime *responds* to the presence of massive objects, causing objects to travel along curves rather than straight lines.

In this chapter we will introduce the notation and necessary mathematics used to describe, understand and obtain an intuitive idea of the modern view of gravity: tensor calculus, curvature and General Relativity. We then apply these tools to cosmology, motivating why the dark sector has been introduced and finally review modified gravity theories. No claim of uniqueness is laid to the content in this chapter. There are many classic texts and reviews on gravity and cosmology [1–11]; we have selected a few to review to enable a relatively self-contained understanding of gravity to be obtained by reading of this chapter.

II. TENSOR CALCULUS, THE METRIC AND CURVATURE

Modern gravitational theories rely on a mathematical framework known as differential geometry; our aim is to provide an illustration of the mathematical construction which is used to model our universe and not of the technical details of differential geometry (for that, see e.g. [5, 12]). A manifold can be thought of as a smooth collection of points, where each point describes an "event" in the universe (an event is a point which has a unique "time" and "space" coordinate). These events may be connected by smooth curves. Objects in the universe travel along these curves. Some geometrical structure in the universe may provide special sets of curves which are unique to a particular universe – if two universes contain different material substances then the "special sets of curves" will be different in these two universes. If, by experimental observation of the universe we inhabit, one can obtain information about these paths then one can determine the underlying geometrical structure of our universe.

A. Notation, coordinates and tensors

An event on a four dimensional manifold requires four numbers to be given to uniquely identify that event; these four numbers are collected and arranged into the components of the contravariant position vector:

$$x^\mu = (x^0, x^1, x^2, x^2). \tag{2.1}$$

In the Cartesian basis the components of this vector are the time coordinate, $x^0 = t$, and the spatial coordinates, $x^i = (x^1, x^2, x^3) = (x, y, z)$. Differentiation of a function Z with respect to the spacetime coordinate x^μ is written as

$$\partial_\mu Z \equiv \frac{\partial Z}{\partial x^\mu}. \tag{2.2}$$

Performing a coordinate transformation finds the values of the coordinates at that event, from a different reference frame; this different frame could correspond to an observer that is accelerating relative to the first observer, or where the second observer is at a different point in a gravitational potential. The coordinate transformation is

$$x^\mu \longrightarrow x'^\mu = x'^\mu(x^\nu). \tag{2.3}$$

The Jacobian $J^\mu{}_\nu$, and its inverse, $J_\nu{}^\mu$, is associated with a given coordinate transformation and is constructed via

$$J^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}, \qquad J_\nu{}^\mu \equiv \frac{\partial x^\mu}{\partial x'^\nu}, \qquad J^\mu{}_\alpha J_\nu{}^\alpha \equiv \delta^\mu{}_\nu. \tag{2.4}$$

Tensors are defined according to the way in which they transform under the coordinate transformation. The rank of a tensor is given by the number of indices in a particular location and is written as $\binom{n}{m}$. A scalar should be thought of as a tensor of rank-0 and a vector as a tensor of rank-1. A tensor of rank $\binom{n}{m}$ has n "upper" contravariant indices and m "lower" covariant indices, and transforms as

$$A'^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m} = \left(\prod_{i=1}^{n}\prod_{j=1}^{m} J^{\mu_i}{}_{\alpha_i} J_{\nu_j}{}^{\beta_j}\right) A^{\alpha_1\alpha_2\ldots\alpha_n}{}_{\beta_1\beta_2\ldots\beta_m}. \tag{2.5}$$

For example, a rank-2 contravariant tensor would look like $A^{\mu\nu}$, and a rank-2 covariant tensor like $A_{\mu\nu}$, and transform as

$$A'^{\mu\nu} = J^\mu{}_\alpha J^\nu{}_\beta A^{\alpha\beta}, \qquad A'_{\mu\nu} = J_\mu{}^\alpha J_\nu{}^\beta A_{\alpha\beta}, \tag{2.6}$$

which is invariant under coordinate transformation. Their contraction forms a scalar,

$$\breve{A}' \equiv A'^{\mu\nu} A'_{\mu\nu} = J^\mu{}_\alpha J^\nu{}_\beta J_\mu{}^\rho J_\nu{}^\sigma A^{\alpha\beta} A_{\rho\sigma} = A^{\alpha\beta} A_{\alpha\beta} \equiv \breve{A}. \tag{2.7}$$

Let us consider the partial derivative operator acting upon the components of a vector, $\partial_\nu A^\mu$. Performing a coordinate transformation yields

$$\begin{aligned} \partial'_\nu A'^\mu &= J_\nu{}^\alpha \partial_\alpha (J^\mu{}_\beta A^\beta) \\ &= J_\nu{}^\alpha J^\mu{}_\beta \partial_\alpha A^\beta + A^\beta J_\nu{}^\alpha \partial_\alpha J^\mu{}_\beta. \end{aligned} \tag{2.8}$$

The final term on the right hand side reveals that $\partial_\nu A^\mu$ does not transform as a $\binom{1}{1}$ tensor, as one may have thought. The "physical" reason for this is that the derivative operation takes the value of the components of the vector two different locations. Tensorial quantities are only defined at a given location on a manifold. The way of remedying this is to introduce the connection, which we will do in the next section.

The components of a tensor may not be independent. For example, if $A_{\mu\nu} = A_{\nu\mu}$ then we say that $A_{\mu\nu}$ is a *symmetric* tensor, and if $B_{\mu\nu} = -B_{\nu\mu}$ then $B_{\mu\nu}$ is *anti-symmetric.* Symmetric and anti-symmetric tensors can be formed from general tensors:

$$S_{\mu\nu} = T_{(\mu\nu)} \equiv \frac{1}{2}\big(T_{\mu\nu} + T_{\nu\mu}\big) = S_{\nu\mu}, \quad A_{\mu\nu} = T_{[\mu\nu]} \equiv \frac{1}{2}\big(T_{\mu\nu} - T_{\nu\mu}\big) = -A_{\nu\mu}. \tag{2.9}$$

B. The metric

One of the simplest questions one can ask is: "*what is the distance between two points?*" Pythagoras' theorem enables distances to be computed on a Euclidean manifold. For two points on an N-dimensional flat space, $\mathbb{R}^N$ whose coordinates differ by the infinitesimal amounts $\mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^N$, the square of the distance between the two points is given by the sum of the squares of the incremental variations in each of the coordinates:

$$\mathrm{d}\ell^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + \ldots + (\mathrm{d}x^N)^2 = \sum_{i,j=1}^{N} \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j = \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{2.10}$$

The first equality is just the usual statement of Pythagoras' theorem; to be able to write the second equality the Kronecker-delta, δ_{ij}, is introduced, which enables Pythagoras' theorem to be more compactly written down; the final equality follows by using the "Einstein summation convention" where repeated indices are summed over. The Kronecker-delta is an identity matrix, whose components are given by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases} \tag{2.11}$$

The quantity $\mathrm{d}\ell^2$ is interpreted as the elemental distance between two infinitesimally separated points, and by introducing the quantity δ_{ij} one becomes able to compactly write this distance down. In this example of a Euclidean space, the quantity δ_{ij} plays the role of the metric, and it is constant throughout the space; this property need not be true in general. An important feature of the Kronecker-delta is that its determinant is positive definite; a manifold endowed with a metric whose determinant is positive-definite is called a Riemannian manifold. The metric defines the geometry of a space.

The Minkowski metric is the simplest example of a metric which can be used to incorporate the time coordinate into the distance measure. The components of the Minkowski metric are given by

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1), \tag{2.12}$$

and the distance measure is

$$\mathrm{d}s^2 = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{2.13}$$

The Minkowksi metric $\eta_{\mu\nu}$ is, like the Kronecker-delta, constant throughout spacetime, but its determinant is negative-definite; a manifold endowed with a metric whose determinant is negative-definite is called a pseudo-Riemannian manifold.

To define distances on a general (possibly curved) manifold, the manifold is endowed with a symmetric rank-2 tensor, the metric. The components of the metric are written as $g_{\mu\nu}$, and their values may depend upon the spacetime location at which they are evaluated: $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$. The metric is used to write down the infinitesimal interval between two spacetime events,

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{2.14}$$

The line element (2.14) is invariant under coordinate transformations (2.3). The norm of a 4-vector is defined

$$\mathsf{A} \cdot \mathsf{A} \equiv g_{\mu\nu} A^\mu A^\nu = A^\mu A_\mu. \tag{2.15}$$

If $A^\mu A_\mu < 0$ then the vector whose whose components are A^μ is called *time-like*, if $A^\mu A_\mu = 0$ then A^μ is *null* and if $A^\mu A_\mu > 0$ then A^μ is *space-like*. The components of the inverse metric are $g^{\mu\nu}$, and is defined so that $g^{\mu\nu} g_{\nu\alpha} = \delta^\mu{}_\alpha$. The metric is used to lower contravariant indices into covariant indices and the inverse metric is used to raise covariant indices into contravariant indices. For example,

$$g_{\mu\nu} A^{\mu\alpha\beta} = A_\nu{}^{\alpha\beta}, \qquad g^{\mu\nu} B_{\mu\alpha\beta} = B^\nu{}_{\alpha\beta}. \tag{2.16}$$

The ordering of the indices is preserved by index raising and lowering.

When the components of the metric depend upon the spacetime coordinates, the derivative of the metric with respect to the spacetime coordinates does not vanish:

$$g_{\mu\nu} = g_{\mu\nu}(x^\alpha) \qquad \Rightarrow \qquad \partial_\alpha g_{\mu\nu} \neq 0. \tag{2.17}$$

When this is the case, the operations of partial differentiation and index raising/lowering do not commute. For example,

$$\partial_\mu A^\alpha = \partial_\mu (g^{\alpha\beta} A_\beta) = g^{\alpha\beta} \partial_\mu A_\beta + A_\beta \partial_\mu g^{\alpha\beta}. \tag{2.18}$$

Obviously, the final term vanishes in a Minkowski spacetime, since there $\partial_\alpha \eta_{\mu\nu} = 0$. In the general case where the components of the metric vary throughout the spacetime manifold, $\partial_\alpha g_{\mu\nu} \neq 0$, a new type of differential operator which preserves the metric is defined. That is, in addition to using the partial derivative ∂, we introduce the *covariant derivative* ∇, such that

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\beta} - \Gamma^\lambda{}_{\alpha\nu} g_{\mu\lambda} = 0. \tag{2.19}$$

This is called the metricity condition and enables the operation of covariant differentiation and index raising/lowering to commute: $\nabla_\mu A^\alpha = \nabla_\mu (g^{\alpha\beta} A_\beta) = g^{\alpha\beta} \nabla_\mu A_\beta$. The covariant derivative of a tensor transforms as a tensor under coordinate transformations (whereas the partial derivative of a tensor does not transform as a tensor). Using this information, one is able to construct the covariant derivative of contravariant and covariant vectors,

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\alpha} A^\alpha, \qquad \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha{}_{\mu\nu} A_\alpha, \tag{2.20a}$$

$$\nabla_\alpha B_{\mu\nu} = \partial_\alpha B_{\mu\nu} - \Gamma^\beta{}_{\alpha\mu} B_{\beta\nu} - \Gamma^\beta{}_{\alpha\nu} B_{\mu\beta}. \tag{2.20b}$$

This can be extended to the covariant derivative of mixed rank tensors,

$$\nabla_\mu T^\alpha{}_\beta = \partial_\mu T^\alpha{}_\beta + \Gamma^\alpha{}_{\mu\nu} T^\nu{}_\beta - \Gamma^\nu{}_{\mu\beta} T^\alpha{}_\nu. \tag{2.20c}$$

These covariant derivatives transform as tensors by construction.

We now ask: "*what is the shortest distance between two points?*". We use the notion of distance, as defined using the metric, to compute the "shortest" distance between two points on a spacetime manifold of arbitrary geometry. We join the two points by a trajectory, where along that trajectory the spacetime coordinates have the values X^μ. By integrating the infinitesimal path length $\mathrm{d}s$ (2.14) along the length of the trajectory the total length of the trajectory is obtained, through a spacetime having a given metric:

$$S = \int \mathrm{d}s = \int \sqrt{g_{\mu\nu} \mathrm{d}X^\mu \mathrm{d}X^\nu}. \tag{2.21}$$

We like to know what the values of the X^μ are along the trajectory. Of course, along an arbitrary trajectory the coordinates X^μ are arbitrary. We can however pick a particular trajectory from the set of all possible trajectories, where this particular trajectory is the shortest possible line joining any two points. What this means is that the path length is extremized with respect to variations in the coordinates that lie on this particular trajectory,

$$\frac{\delta S}{\delta X^\mu} = 0. \tag{2.22}$$

To calculate the values of the coordinates along this trajectory we introduce an affine parameter λ along the curve, so that $X^\mu = X^\mu(\lambda)$. This allows the path length (2.21) to be written as

$$S = \int \mathrm{d}\lambda \sqrt{g_{\mu\nu}(X^\alpha)\dot{X}^\mu \dot{X}^\nu}\,, \tag{2.23}$$

where an overdot denotes derivative with respect to the affine parameter, $\dot{X}^\mu \equiv \mathrm{d}X^\mu/\mathrm{d}\lambda$. Treating the integrand in (2.23) as a Lagrangian, the principle of least action can be used to obtain the Euler-Lagrange equation whose solution (i.e. the $X^\mu(\lambda)$) extremizes the path length (2.21). The Euler-Lagrange equation one obtains is

$$\ddot{X}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{X}^\alpha \dot{X}^\beta = 0, \tag{2.24}$$

where $\Gamma^\mu{}_{\alpha\beta}$ are the components of the Christoffel symbol and are given by

$$\Gamma^\mu{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right) = \Gamma^\mu{}_{\beta\alpha}. \tag{2.25}$$

Equation (2.24) is called the *equation of an affinely parameterized geodesic*. It is important to realise that the $\Gamma^\mu{}_{\alpha\beta}$ do not form the components of a tensor. Notice that for flat space, $\Gamma^\alpha{}_{\mu\nu} = 0$, so that the geodesic (2.24) becomes $\ddot{X}^\mu = 0$. What this means is that geodesics in flat space are straight lines.

The geodesic equation (2.24) can only be solved once the components of the metric tensor, $g_{\mu\nu}$, are known. To know what the components $g_{\mu\nu}$ are for a particular spacetime, gravitational field equations are required which we will discuss shortly.

C. Curvature

The manifolds discussed are, in general, curved. The curvature of a manifold is characterized by the second derivative of the fundamental object defining the manifold, which is the metric. A manifold whose metric has non-zero second derivatives is curved. One usually imagines curvature of a surface that is embedded in a higher-dimensional space. This is *extrinsic* curvature. It is also possible to develop the notion of *intrinsic* curvature without needing to refer to some higher-dimensional space. We will begin our discussion by constructing the intrinsic curvature tensors and then we construct extrinsic curvature tensors.

To quantify intrinsic curvature, tensors are constructed from the second derivatives of the metric; the so-called curvature tensors. The first tensor is known as the Riemann tensor, and arises by considering the commutator of covariant derivatives. One is able to obtain the Ricci identity

$$\big(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\big) A^\alpha = R^\alpha{}_{\beta\mu\nu} A^\beta, \tag{2.26}$$

where the Riemann tensor, $R^\alpha{}_{\beta\mu\nu}$, is given by

$$R^\alpha{}_{\beta\mu\nu} \equiv 2\partial_{[\mu}\Gamma^\alpha{}_{\nu]\beta} + 2\Gamma^\alpha{}_{\rho[\mu}\Gamma^\rho{}_{\nu]\beta}. \tag{2.27}$$

In flat space the Christoffel symbols vanish so that covariant derivatives reduce to partial derivatives, which commute and so $R^\alpha{}_{\beta\mu\nu} = 0$ for flat space. The Ricci tensor is formed by setting the first and third indices of the Riemann tensor to be equal,

$$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu}, \tag{2.28}$$

The contracted Ricci identity is

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A^\mu = R_{\alpha\nu} A^\alpha. \tag{2.29}$$

Finally, by contracting the Ricci tensor one obtains the Ricci scalar,

$$R \equiv R^\mu{}_\mu. \tag{2.30}$$

The Ricci tensor and scalar are combined to produce the Einstein tensor,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.31}$$

whose indices are symmetric, $G_{\mu\nu} = G_{(\mu\nu)}$. One can use the above definitions to show that the Einstein tensor satisfies a conservation equation,

$$\nabla_\mu G^{\mu\nu} = 0, \tag{2.32}$$

which is called the Bianchi identity.

The components of a tensor change under a coordinate transformation, but the value of a scalar does not. This means that, for example, the values of the components of the Ricci tensor $R_{\mu\nu}$ change, but the value of the Ricci scalar does not. With this in mind, we write down three curvature invariants,

$$R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}, \qquad R^{\mu\nu} R_{\mu\nu}, \qquad R, \tag{2.33}$$

whose values remain the same regardless of coordinate system. Usefully, any singularities present in these curvature invariants are intrinsic to the manifold, and are unable to be removed by coordinate transformation.

Whilst the theories of gravity we will consider are covariant (i.e. space and time are on an "equal footing"), it is useful to write down expressions which give 3D "space" and "time" some meaning in a covariant way. We imagine 3D sheets as being embeded in the 4D universe, where to travel along a given sheet means to travel through space, and to different sheets means to travel through time. This gives rise to the notion of the extrinsic curvature of the 3D sheets, due to their embedding in a higher dimensional space. We foliate the 4D universe by 3D sheets, where there is a metric $\gamma_{\mu\nu}$ that exists on these sheets and a time-like unit vector u^μ is everywhere orthogonal to these sheets. The metric and curvature tensors are rewritten by imposing the following structure

$$g_{\mu\nu} = \gamma_{\mu\nu} - u_\mu u_\nu, \qquad u^\mu u_\mu = -1, \qquad u^\mu \gamma_{\mu\nu} = 0. \tag{2.34}$$

The vector u^μ is a time-like geodesic, and so

$$u^\mu \nabla_\mu u_\nu = 0. \tag{2.35}$$

We then define the extrinsic curvature tensor

$$K_{\mu\nu} = \nabla_\mu u_\nu, \tag{2.36}$$

and, by using (2.34, 2.35), can be shown to satisfy

$$u^\mu K_{\mu\nu} = 0 \qquad \Longrightarrow \qquad K_{\mu\nu} = \gamma^\alpha{}_\mu \nabla_\alpha u_\nu. \tag{2.37}$$

The trace of the extrinsic curvature, $K \equiv K^\mu{}_\mu = \gamma^{\mu\nu} K_{\mu\nu}$ will correspond to the curvature scalar of the 3D sheets due to their embedding in a higher dimensional space. Differentiation along the time-like direction and differentiation confined to the 3D sheets are defined respectively as

$$\dot{A}^\mu \equiv u^\alpha \nabla_\alpha A^\mu, \qquad \bar{\nabla}_\mu A^\nu \equiv \gamma^\alpha{}_\mu \gamma^\nu{}_\beta \nabla_\alpha A^\beta. \tag{2.38}$$

With this definition we find that $\bar{\nabla}_\mu$ is the covariant derivative which preserves the 3D metric,

$$\bar{\nabla}_\mu \gamma_{\alpha\beta} = 0. \tag{2.39}$$

By combining the various definitions given above, we compute the anti-symmetric sheet-confined derivative of a vector which it itself confined to the 3D sheet (i.e. $u_\mu V^\mu = 0$),

$$2\bar{\nabla}_{[\alpha}\bar{\nabla}_{\beta]} V_\nu = -\gamma^\pi{}_\alpha \gamma^\rho{}_\beta \gamma^\sigma{}_\nu R^\epsilon{}_{\sigma\pi\rho} V_\epsilon + (K_{\beta\nu} K^\epsilon{}_\alpha - K_{\alpha\nu} K^\epsilon{}_\beta) V_\epsilon. \tag{2.40}$$

The left-hand-side of this expression is defined to be the Riemann tensor on the 3D sheet, $2\bar{\nabla}_{[\alpha}\bar{\nabla}_{\beta]} V_\nu \equiv -{}^{(3)}R^\epsilon{}_{\nu\alpha\beta} V_\epsilon$. Hence, a relationship is obtained between the Riemann tensor on the 3D sheet, in the 4D manifold in which the sheet is embedded and the extrinsic curvature tensors:

$${}^{(3)}R^\epsilon{}_{\nu\alpha\beta} = \gamma^\pi{}_\alpha \gamma^\rho{}_\beta \gamma^\sigma{}_\nu R^\epsilon{}_{\sigma\pi\rho} + K_{\alpha\nu} K^\epsilon{}_\beta - K_{\beta\nu} K^\epsilon{}_\alpha. \tag{2.41}$$

By contraction, we obtain the 3D Ricci tensor,

$${}^{(3)}R_{\nu\beta} = \gamma^\rho{}_\beta \gamma^\sigma{}_\nu R_{\sigma\rho} + \gamma^\rho{}_\beta \gamma^\sigma{}_\nu u^\pi u_\epsilon R^\epsilon{}_{\sigma\pi\rho} + K_{\epsilon\nu} K^\epsilon{}_\beta - K_{\beta\nu} K, \tag{2.42}$$

and computing ${}^{(3)}R \equiv g^{\mu\nu\,(3)}R_{\mu\nu}$ we obtain the 3D Ricci scalar,

$${}^{(3)}R + K^2 - K^{\mu\nu} K_{\mu\nu} = R + 2u^\mu u^\nu R_{\mu\nu} = 2u^\mu u^\nu G_{\mu\nu}. \tag{2.43a}$$

Similarly, one can compute

$$\bar{\nabla}_\mu K^\mu{}_\alpha - \bar{\nabla}_\alpha K = \gamma^\mu{}_\alpha u^\nu R_{\mu\nu} \tag{2.43b}$$

The equations (2.43) are known as the Gauss-Codacci relations, and enable the geometry of 3D sheets to be determined by their embedding in a 4D manifold.

III. GENERAL RELATIVITY

General Relativity (GR) is arguably one of the greatest advances in theoretical physics from the past century. Although the mathematical formalism existed well before Albert Einstein laid out what we now call GR, the key insight which Einstein had was to understand that the geometry of the universe is determined by, and responds to, the gravitating content of the universe. Einstein constructed a set of dynamical rules for spacetime with arbitrary geometry; these rules are now what we call *Einstein's field equations of General Relativity.* Gravity is interpreted as the manifestation of the location-dependance of the metric. The field equations determine the values of the components of the metric of a spacetime for some known content. Once the metric is known one can begin to compute geodesics in the spacetime: these are the paths that bundles of light rays travel along or the orbits that planets trace out. This enables GR, as a gravitational theory, to predict directly observable quantities. This also applies to the entire universe: Einstein's field equations allow the metric for the entire universe to be computed once the content of the universe is known.

Einstein's field equations of GR provide a specific way in which the metric is determined from the content of the spacetime. The information about the content is contained within the energy-momentum tensor $T_{\mu\nu}$. Einstein's field equations of GR relate $G_{\mu\nu}$ and $T_{\mu\nu}$ linearly,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3.1}$$

where the constant of proportionality, $8\pi G$, is determined by comparing the Newtonian limit of the GR field equations with the Newtonian field equations. The field equations (3.1) are second order evolution equations for the metric, and they are sourced by the energy-momentum tensor. Because the Einstein tensor satisifies a Bianchi identity, (2.32), so too must the energy-momentum tensor,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.2}$$

This can be interpreted as either a constraint equation or an evolution equation for the matter content. In the absence of gravitational fields, the conservation equation reads $\partial_\mu T^{\mu\nu} = 0$, but in the presence of gravitational fields,

$$\partial_\mu T^{\mu\nu} = -\Gamma^\mu{}_{\mu\alpha} T^{\alpha\nu} - \Gamma^\nu{}_{\mu\alpha} T^{\mu\alpha}, \tag{3.3}$$

so that gradients of the metric contribute to the dynamics of the fluid. Without knowledge of this sourcing, it would appear that the fluid does not satisfy a conservation equation.

A. Exact analytic solutions to Einstein's field equations

No general solution to Einstein's field equations (3.1) is known to exist. However, solutions are known for rather specific configurations of the content. The simplest solution is for a spacetime which is completely empty (of gravitating matter) is called a vacuum, and whose metric is given by the Minkowski metric,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{3.4}$$

The next simplest metric is that for a spacetime containing a homogeneous and isotropic fluid, whose energy-momentum tensor is of the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + P\gamma_{\mu\nu}, \tag{3.5}$$

where $\rho = \rho(t), P = P(t)$, and is given by Friedmann-Robertson-Walker's solution [13],

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\big(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\big). \tag{3.6}$$

Once distributions of matter which are confined to particular locations are included the solutions become much less attractive. The energy-momentum tensor for a single stationary mass can be written as $T^\mu{}_\nu = M\delta^{(3)}(\mathbf{r})u^\mu u_\nu$. The metric of a spacetime which asymptotes to vacuum at infinity which contains a single stationary black hole of mass M at the origin, is given by Schwarzschild's solution [14],

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{3.7}$$

where $\mathrm{d}\Omega^2 \equiv (\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\phi^2)$ is the solid angle element. The metric for a spacetime containing a stationary black hole of mass M in a universe containing a cosmological constant Λ is given by Kottler's solution [15],

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{3.8}$$

also called a Schwarzschild de-Sitter metric. A single black hole of mass M immersed in a homogeneous isotropic fluid has a metric which is given by McVittie's solution [16, 17],

$$\mathrm{d}s^2 = -\left(\frac{2a(t)r - M}{2a(t)r + M}\right)^2\mathrm{d}t^2 + a^2(t)\left(1 + \frac{M}{2a(t)r}\right)^4\big(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\big). \tag{3.9}$$

Kerr's solution [18] provides the metric for a spacetime containing a single black hole of mass M rotating with angular momentum $J = M\omega$,

$$\begin{aligned} \mathrm{d}s^2 \;=\; & -\frac{r^2 - 2Mr + \omega^2 - \omega^2\sin^2\theta}{r^2 + \omega^2\cos^2\theta}\mathrm{d}t^2 - \frac{4M\omega r\sin^2\theta}{r^2 + \omega^2\cos^2\theta}\mathrm{d}\phi\mathrm{d}t \\ & + \frac{\sin^2\theta}{r^2 + \omega^2\cos^2\theta}\left[\big(r^2 + \omega^2\big)^2 - \omega^2\sin^2\theta\,\big(r^2 - 2Mr + \omega^2\big)\right]\mathrm{d}\phi^2 \\ & + \big(r^2 + \omega^2\cos^2\theta\big)\left[\frac{\mathrm{d}r^2}{r^2 - 2Mr + \omega^2} + \mathrm{d}\theta^2\right]. \end{aligned} \tag{3.10}$$

B. Perturbative gravity

As remarked above, it is not known how to solve the gravitational field equations in general. This is largely because they are highly complicated non-linear differential equations. The problem can be simplified by using perturbation theory so that the gravitational field equations become linear in the metric. Suppose, for example, that the metric $\bar{g}_{\mu\nu}$ is known for some simple physical system. If the physical system of interest has a metric given by $g_{\mu\nu}$, and can be considered as a small deviation about that simple system, $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, then perturbation theory can be used to obtain the linearized version of the gravitational field equations. We will now provide formulae for the linearized gravitational field equations.

The background value of the metric is written with an overline, $\bar{g}_{\mu\nu}$, and the perturbation to that background is denoted by $h_{\mu\nu}$. The metric thus written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \bar{g}^{\mu\nu} + h^{\mu\nu}. \tag{3.11}$$

All quantities calculated from the metric will also be perturbed. For example, inserting (3.11) into the Christoffel symbol (2.25) we find

$$\Gamma^{\mu}{}_{\alpha\beta} = \bar{\Gamma}^{\mu}{}_{\alpha\beta} + \delta\Gamma^{\mu}{}_{\alpha\beta}, \tag{3.12}$$

where the background and perturbed contributions to the overall Christoffel symbol are given by

$$\bar{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{1}{2}\bar{g}^{\mu\nu}\big(\partial_\alpha \bar{g}_{\beta\nu} + \partial_\beta \bar{g}_{\alpha\nu} - \partial_\nu \bar{g}_{\alpha\beta}\big), \tag{3.13a}$$

$$\delta\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}\bar{g}^{\mu\nu}\big(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\big) + \bar{\Gamma}^{\rho}{}_{\alpha\beta} h^{\mu}{}_{\rho}. \tag{3.13b}$$

The formula for the perturbed Christoffel symbol (3.13b) can be shown to be equivalent to

$$\delta\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}\bar{g}^{\mu\nu}\big(\nabla_\alpha h_{\beta\nu} + \nabla_\beta h_{\alpha\nu} - \nabla_\nu h_{\alpha\beta}\big), \tag{3.14}$$

where ∇_α is the covariant derivative with respect to the background metric $\bar{g}_{\mu\nu}$. This way of writing the perturbed Christoffel symbol is more convenient than (3.13b). The perturbed Christoffel symbol can also be written as

$$\delta\Gamma^{\mu}{}_{\alpha\beta} = \big(\bar{g}^{\mu\pi}\delta^{\rho}{}_{(\alpha}\delta^{\sigma}{}_{\beta)} - \tfrac{1}{2}\bar{g}^{\mu\rho}\delta^{\sigma}{}_{\alpha}\delta^{\pi}{}_{\beta}\big)\nabla_\rho h_{\sigma\pi}. \tag{3.15}$$

After similar manipulations one obtains the perturbation to the Riemann tensor,

$$\delta R^{\alpha}{}_{\mu\beta\nu} = \big(g^{\alpha\rho}\delta^{\sigma}{}_{\mu}\delta^{[\pi}{}_{\beta}\delta^{\xi]}{}_{\nu} + g^{\alpha\pi}\delta^{\sigma}{}_{\mu}\delta^{[\rho}{}_{\nu}\delta^{\xi]}{}_{\beta} + g^{\alpha\pi}\delta^{\rho}{}_{\mu}\delta^{[\sigma}{}_{\nu}\delta^{\xi]}{}_{\beta}\big)\nabla_\xi\nabla_\rho h_{\sigma\pi}. \tag{3.16}$$

In summary, the perturbations to the Christoffel symbol, Ricci tensor, Ricci scalar and Einstein tensor are given by

$$\delta\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\alpha\beta}\big(\nabla_\mu h_{\nu\beta} + \nabla_\nu h_{\mu\beta} - \nabla_\beta h_{\mu\nu}\big), \tag{3.17a}$$

$$\delta R_{\mu\nu} = \nabla^\alpha \nabla_{(\mu} h_{\nu)\alpha} - \frac{1}{2}\big(\Box h_{\mu\nu} + \nabla_\mu\nabla_\nu h\big), \tag{3.17b}$$

$$\delta R = \nabla^\mu\nabla^\nu h_{\mu\nu} - \Box h - R^{\mu\nu}h_{\mu\nu}, \tag{3.17c}$$

$$\begin{aligned} 2\delta G_{\mu\nu} \;=\; & \nabla^\alpha\nabla_\mu h_{\nu\alpha} + \nabla^\alpha\nabla_\nu h_{\mu\alpha} - \Box h_{\mu\nu} + \bar{g}_{\mu\nu}\Box h - \nabla_\mu\nabla_\nu h \\ & -\bar{g}_{\mu\nu}\nabla^\alpha\nabla^\beta h_{\alpha\beta} - R h_{\mu\nu} + \bar{g}_{\mu\nu}R^{\alpha\beta}h_{\alpha\beta}, \end{aligned} \tag{3.17d}$$

where we wrote $h = h^{\mu}{}_{\mu} = \bar{g}^{\mu\nu}h_{\mu\nu}$. The covariant derivative, ∇_μ, Ricci tensor and scalar $R_{\mu\nu}, R$ are those of the background spacetime. For perturbations about Minkowski spacetime, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, the perturbed field equations of General Relativity yield

$$\begin{aligned} 2\delta G_{\mu\nu} \;&=\; \partial^\alpha\partial_\mu h_{\nu\alpha} + \partial^\alpha\partial_\nu h_{\mu\alpha} - \Box h_{\mu\nu} + \eta_{\mu\nu}\Box h - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} \\ &=\; 16\pi G\delta T_{\mu\nu}. \end{aligned} \tag{3.18}$$

C. Field equations from an action

The formulation of the gravitational field equations we have presented has been geometrical: the Einstein tensor was constructed from various combinations of derivatives of the metric, and was equated to the energy-momentum tensor. The field equations can be derived in another way, using the principle of least action. This strategy is algorithmic and it becomes obvious as to how we can construct "modified gravity" theories. We start from the Einstein-Hilbert action

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[R - 16\pi G\mathcal{L}_{\mathrm{m}}\Big], \tag{3.19}$$

where g is the determinant of the metric tensor, R is the Ricci scalar and $\mathcal{L}_{\mathrm{m}}$ is the Lagrangian density of all sources of energy and momentum. Applying the principle of least action to (3.19) yields the field equations of General Relativity. The field equations (of the metric) are found by extremizing the variation of the action with respect to variations in the metric,

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0. \tag{3.20}$$

Varying the action (3.19) yields

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R - 16\pi G\mathcal{L}_{\mathrm{m}}}{\sqrt{-g}}\delta(\sqrt{-g}) + \delta R - 16\pi G\delta\mathcal{L}_{\mathrm{m}}\right]. \tag{3.21}$$

To go further, we need the following results:

$$\frac{1}{\sqrt{-g}}\delta\sqrt{-g} = -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}, \qquad \delta R = \Big(R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\Big)\delta g^{\mu\nu}. \tag{3.22}$$

Using (3.22) in (3.21) provides a formula for the variation of the Einstein-Hilbert action (3.19),

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left[G_{\mu\nu}\delta g^{\mu\nu} + \big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\big)\delta g^{\mu\nu} - 16\pi G\frac{1}{\sqrt{-g}}\delta\big(\sqrt{-g}\mathcal{L}_{\mathrm{m}}\big)\right], \tag{3.23}$$

where the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is identified. The second term in the integrand is a total derivative, which can be rewritten as a surface integral and neglected (in the following section we will show how to deal with this surface integral). In theories where the Ricci scalar does not appear linearly these terms do not correspond to total derivatives, cannot be removed and will therefore represent a modification to the field equations. By setting the variation of the action with respect to the variation of the metric to zero, $\delta S/\delta g^{\mu\nu} = 0$, and assuming that the variation of the metric vanishes on the boundary, we obtain from (3.23) the field equations for the metric

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3.24}$$

where the energy-momentum tensor is defined as

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\big(\sqrt{-g}\mathcal{L}_{\mathrm{m}}\big). \tag{3.25}$$

Hence, we observe that by applying the variational principle (3.20) to the Einstein-Hilbert action (3.19) we obtained the Einstein field equations (3.24). By obtaining the gravitational field equations from the principle of least action of a Lagrangian, we gain some intuition that the values of the metric which solve the field equations (3.24) extremize the value of some global measure.

D. Review of the Gibbons-Hawking term

In the discussion preceding (3.24) it was mentioned that the surface term requires special attention: here we show how to deal with the surface term by reviewing the Gibbons-Hawking term. We begin with a discussion from classical mechanics of actions containing a total derivative [19] which will build up some intuition of total derivatives before moving on to discuss the total derivative in GR.

The action of a Lagrangian containing at most first derivatives of the generalized coordinate is

$$S = \int_{t_1}^{t_2} \mathrm{d}t \, L(q, \dot{q}), \tag{3.26}$$

and can be varied to yield

$$\begin{aligned} \delta S = \int_{t_1}^{t_2} \mathrm{d}t \left(\delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} \right) &= \int_{t_1}^{t_2} \mathrm{d}t \left\{ \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \delta q + \frac{\mathrm{d}}{\mathrm{d}t} \left(\delta q \frac{\partial L}{\partial \dot{q}} \right) \right\} \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left\{ \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} \\ &\qquad + \delta q \frac{\partial L}{\partial \dot{q}} \bigg|_{t_1}^{t_2} . \end{aligned} \tag{3.27}$$

To be able to use the variational principle to obtain the equations of motion *boundary data*, prescribing the values of the variations at the endpoints of the trajectory, $\delta q(t_1), \delta q(t_2)$, must be specified. These variations are usually taken to vanish. Let us now construct a Lagrangian by adding a total derivative, L_2, to a pre-existing theory, L_1:

$$L = L_1 + L_2 = -\frac{1}{2} q \ddot{q}, \qquad L_1 \equiv \frac{1}{2} \dot{q}^2, \quad L_2 \equiv -\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} q \dot{q} \right). \tag{3.28}$$

The variation of the Lagrangian L yields

$$\delta L = \ddot{q} \delta q + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \big(\dot{q} \delta q - q \delta \dot{q} \big). \tag{3.29}$$

Notice that the equation of motion of L and of L_1 are identical ($\ddot{q} = 0$): the addition of L_2 does not affect the dynamical equations of motion because L_2 is a total derivative and only contributes on the boundary. To obtain the equations of motion for L four pieces of boundary data are required, (i.e. the values of $\delta q(t_1), \delta q(t_2)$ and $\delta \dot{q}(t_1), \delta \dot{q}(t_2)$), whilst for the theory with L_1 only two pieces of boundary data are required (the values of the variation at the endpoints, $\delta q(t_1), \delta q(t_2)$.

The point is: by adding on a total derivative (which does not change the equations of motion) the amount of data that is required to be specified on the boundary is changed, and this can be thought of as introducing an "inconsistency". To bring the theory back into "consistency" we modify the theory by adding on a boundary term to the Lagrangian which will kill off the offending term. So, the theory L must be modified to

$$L \to \check{L} = L + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}q\dot{q}\right). \tag{3.30}$$

The Lagrangian L can be thought of as being analogous to the Einstein-Hilbert action, and the boundary term $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}q\dot{q}\right)$ can be thought of as being analogous to the Gibbons-Hawking term, as we will now show.

Our discussion now takes inspiration from a number of sources [1, 20–23]. We will start by showing what the problem is with only specifying the Einstein-Hilbert action

$$S = \int \mathrm{d}^4x \sqrt{-g}\, R = \int_M R, \tag{3.31}$$

and then go on to show the popular way to resolve the problem. Without removing any terms, the variation of the Einstein-Hilbert action yields

$$\delta S = \int_M \left[G_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right], \tag{3.32}$$

where one can obtain

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\alpha v^\alpha, \qquad v^\alpha \equiv g_{\mu\nu}\big(\nabla^\alpha \delta g^{\mu\nu} - \nabla^\nu \delta g^{\alpha\mu}\big). \tag{3.33}$$

Hence, the variation of the Einstein-Hilbert action (3.32) using (3.33) can be written as

$$\delta S = \int_M \left[G_{\mu\nu}\delta g^{\mu\nu} + \nabla_\alpha v^\alpha\right]. \tag{3.34}$$

By applying the divergence theorem to the second term, this becomes

$$\delta S = \int_M G_{\mu\nu}\delta g^{\mu\nu} + \int_{\partial M} n_\alpha v^\alpha, \tag{3.35}$$

where n_α is a unit normal to the boundary. Within the surface integral we have

$$n_\alpha v^\alpha = n_\alpha g_{\mu\nu}\big(\nabla^\alpha \delta g^{\mu\nu} - \nabla^\nu \delta g^{\alpha\mu}\big). \tag{3.36}$$

Therefore, after varying the metric in the Einstein-Hilbert action, there is a "surface" contribution. The metric on ∂M is given by $\gamma_{\mu\nu} = g_{\mu\nu} \pm n_\mu n_\nu$, where n^μ is a normal unit vector to the boundary ∂M. Equation (3.36) can be rewritten as

$$n_\alpha v^\alpha = n_\alpha \gamma_{\mu\nu}(\nabla^\alpha \delta g^{\mu\nu} - \nabla^{(\mu}\delta g^{\nu)\alpha}). \tag{3.37}$$

The point of this was to show that the metric $g_{\mu\nu}$ in (3.36) can be replaced with the induced (boundary) metric $\gamma_{\mu\nu}$ in $n_\alpha v^\alpha$. After lowering the indices appropriately (3.37) is used to obtain the following formula for the variation of the Einstein-Hilbert action (3.35):

$$\delta S \;=\; \delta\int_M R = \int_M G_{\mu\nu}\delta g^{\mu\nu} + \int_{\partial M}\left[n^\alpha \gamma^{\mu\nu}\big(\nabla_{(\mu}\delta g_{\nu)\alpha} - \nabla_\alpha \delta g_{\mu\nu}\big)\right]. \tag{3.38}$$

As it stands, the behaviour of the derivative of $\delta g_{\mu\nu}$ must be specified normal to the surface, $n^\alpha \nabla_\alpha \delta g_{\mu\nu}$, as well as the the transverse derivative $\gamma^{\mu\nu}\nabla_{(\mu}\delta g_{\nu)\alpha}$; note that the transverse derivative of the perturbed metric tells us how the metric varies on the boundary. The requirement of vanishing normal derivative can be removed by introducing an extra term into the action, where the extra term only resides on the boundary and encodes information about the geometry of the boundary.

The Gibbons-Hawking term [20] (sometimes called the Gibbons-Hawking-York term [24]) is an explicit example of a boundary term which can be added to the Einstein-Hilbert action to provide the field equations of General Relativity after employing the variational principle. Without adding the term more stringent conditions must be imposed upon the behaviour of the varied metric at the boundary. The Gibbons-Hawking term is the trace of the extrinsic curvature of the boundary, $K = K^\mu{}_\mu$, where $K_{\mu\nu} \equiv \nabla_\mu n_\nu$. The Einstein-Hilbert action with the Gibbons-Hawking term is given by

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,R + \oint \mathrm{d}^3x\sqrt{-\gamma}\,K = \frac{1}{2}\int_M R + \int_{\partial M} K. \tag{3.39}$$

The variation of (3.39) yields

$$\delta S = \frac{1}{2}\int_M G_{\mu\nu}\delta g^{\mu\nu} + \int_{\partial M}\left(\frac{1}{2}n_\alpha v^\alpha + \delta K + \frac{1}{2}K\gamma^{\mu\nu}\delta\gamma_{\mu\nu}\right), \tag{3.40}$$

where $v_\alpha n^\alpha$ is the surface contribution from the variation of the Ricci scalar, defined in (3.37). With the definitions of the extrinsic curvature tensor and $n^\mu n_\mu = 1$, one can deduce the following identities:

$$\nabla_\alpha\gamma_{\mu\nu} = -2K_{\alpha(\mu}n_{\nu)}, \qquad \delta n_\mu = \frac{1}{2}n_\mu n^\alpha n^\beta \delta g_{\alpha\beta}, \tag{3.41a}$$

$$\delta\gamma_{\mu\nu} = \delta g_{\mu\nu} \pm n_\mu \delta n_\nu \pm n_\nu \delta n_\mu. \tag{3.41b}$$

The variation of the trace $K = \gamma^{\mu\nu}K_{\mu\nu}$ yields

$$\begin{aligned}\delta K &= K_{\mu\nu}\delta\gamma^{\mu\nu} - \gamma^{\mu\nu}n_\alpha\delta\Gamma^\alpha{}_{\mu\nu} + \gamma^{\mu\nu}\nabla_\nu\delta n_\mu \\ &= -K^{\mu\nu}\delta g_{\mu\nu} - \gamma^{\mu\nu}n^\alpha\left(\nabla_{(\mu}\delta g_{\nu)\alpha} - \frac{1}{2}\nabla_\alpha\delta g_{\mu\nu}\right) + \frac{1}{2}Kn^\alpha n^\beta\delta g_{\alpha\beta}.\end{aligned} \tag{3.42}$$

These expressions can be combined and inserted into the integrand of the surface term of (3.40), yielding

$$\begin{aligned}\frac{1}{2}n_\alpha v^\alpha + \delta K + \frac{1}{2}K\gamma^{\mu\nu}\delta\gamma_{\mu\nu} &= -\frac{1}{2}\gamma^{\mu\nu}n^\alpha\nabla_{(\mu}\delta g_{\nu)\alpha} \\ &\quad + \frac{1}{2}(Kn^\mu n^\nu + K\gamma^{\mu\nu} - 2K^{\mu\nu})\delta g_{\mu\nu}.\end{aligned} \tag{3.43}$$

Thus, the variation of the Einstein-Hilbert action with a Gibbons-Hawking term yields

$$2\delta S = \int_M G_{\mu\nu}\delta g^{\mu\nu} - \int_{\partial M}\left(\gamma^{\mu\nu}n^\alpha\nabla_{(\mu}\delta g_{\nu)\alpha} - (Kn^\mu n^\nu + K\gamma^{\mu\nu} - 2K^{\mu\nu})\delta g_{\mu\nu}\right). \tag{3.44}$$

All we have to impose now is that the metric does not vary on the boundary: $\delta g_{\mu\nu}|_{(\partial M)} = 0$. This immediately removes all but the first term in the surface integral. A corollary of this condition is that the transverse derivative term vanishes, which completely removes the surface integral. Thus, after imposing $\delta g_{\mu\nu}|_{(\partial M)} = 0$, the variation of the Einstein-Hilbert action with the Gibbons-Hawking term (3.39) yields

$$\delta S = \frac{1}{2}\int_M G_{\mu\nu}\delta g^{\mu\nu}. \tag{3.45}$$

The data $\delta g_{\mu\nu}|_{(\partial M)} = 0$, which follows from GR with the Gibbons-Hawking term, is a smaller amount of data than that required from GR alone, to produce the same field equations. Most of the time the Gibbons-Hawking term is implicitly assumed to be present when the variational principle is used to compute field equations.

IV. COSMOLOGY

Over the past century the view of the universe that we find ourselves in has exploded in scale. One hundred years ago, humanity thought that the Milky Way Galaxy was alone in the universe. In 1916 Albert Einstein published his theory of gravity, General Relativity, which revolutionized our understanding of the nature of spacetime and provided an explanation of various anomalies observed in the orbit of Mercury. In 1925 came the realization that the observed "nebulae" were in fact galaxies in their own right. In 1929, Edwin Hubble observed the recession speeds of galaxies as a function of their distance away from us, providing the rather startling conclusion that the universe is expanding. Over the next seventy years the existence of dark matter was inferred and the cosmic microwave background was detected. In 1998 it was discovered that the universe is actually apparently accelerating in its expansion.

Cosmology is the grandiose study of the universe as a whole, and is tasked with the attempt to provide an understanding of the origin of the universe, how the constituents of the universe affect its geometry and evolution, and how structures form in the universe. The universe is an incredibly complicated object whose constituents vary in size from the quantum mechanically small to the unimaginably large. At first sight it would appear a fruitless task to attempt to construct a metric for the universe which is capable of incorporating information about objects of such a wide range of scales. Indeed, this task is fruitless. However, if assertions are made about the statistical nature in which the content of the universe is distributed on the largest scales, a metric is able to be constructed and used to solve the field equations. The assertion which makes this possible is simply that we are not in a special place in the universe, meaning that the content of the universe is homogeneous and isotropic on the very largest scales. These are the scales where the individual constituents of the universe (galaxies, clusters etc) become so small they are able to be approximated as being the elements of a homogeneous isotropic fluid. We then speak of a cosmological fluid, where the fluid is a mixture of different components each having different gravitational properties.

Of course, modeling the universe as a homogeneous isotropic fluid is an approximation which is well motivated on statistical grounds. By analogy, a sponge is homogeneous if one zooms out far enough, and one certainly does not care about the motion of individual molecules of water if one wants to understand the gross flow of water. However, if we want to model the universe on slightly smaller scales and understand how structures in the universe form and evolve, the fluid description breaks down and we must provide an alternative description. We now imagine that the localized constituents of the universe are immersed within the cosmological fluid and that the dynamics of the fluid will effect the dynamics of these localized constituents; for instance, a different background dynamic will produce a different distribution of the constituents.

We now provide a brief overview of how the cosmological background is modeled within the framework of General Relativity, and how the different constituents in the universe affect its geometry. We then go on to show how to deal with small perturbations about this background due to structure in the universe.

A. The cosmological background

The cosmological principle states that we do not inhabit a special place in the universe; the implication is that the universe is homogeneous and isotropic. Homogeneity implies an invariance of the metric under translation, and isotropy an invariance under rotation. The Friedmann-Robertson-Walker (FRW) metric is constructed under these principles to describe the background geometry of the universe, and is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\bigg(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\bigg), \tag{4.1}$$

where for simplicity we have assumed that the spatial sections have zero curvature. The only "free" function in this metric is the scale factor, $a(t)$: once that is specified, the global geometry of the universe is specified. In General Relativity, the evolution of the scale factor is set by Einstein's field equations. It is common to write down the FRW metric in conformal time via the coordinate transformation $\mathrm{d}t = a\mathrm{d}\tau$. This provides a metric which is conformally flat, where the conformal factor is the scale factor: $g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$. The FRW metric in conformal time is

$$\mathrm{d}s^2 = a^2(\tau)\bigg(-\mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\bigg). \tag{4.2}$$

The FRW metric (4.1) is used to construct geometrical quantities, such as the Christoffel symbols, ${\Gamma^\alpha}_{\mu\nu}$, the Ricci scalar, R, and the components of the Einstein tensor, ${G^\mu}_\nu$. Using (4.1) the non-zero components of these quantities are

$${\Gamma^i}_{j0} = H{\delta^i}_j, \qquad {\Gamma^0}_{ij} = a^2 H\delta_{ij}, \tag{4.3a}$$

$$R = 6\bigg(H^2 + \frac{\ddot{a}}{a}\bigg), \qquad {G^0}_0 = 3H^2, \qquad {G^i}_j = -\bigg(H^2 + 2\frac{\ddot{a}}{a}\bigg){\delta^i}_j, \tag{4.3b}$$

where the Hubble parameter is defined as $H \equiv \dot{a}/a$, and an overdot denotes derivative with respect to the coordinate time t. One can immediately notice that the Ricci scalar has a singularity at $a(t) = 0$: this is called the big bang. The trace of the extrinsic curvature (2.36) is given by $K = 3H$.

Since the Einstein tensor for the FRW metric is diagonal, so too must be the energy-momentum tensor. The content of the universe is modeled as a perfect fluid, whose energy-momentum tensor is given by

$${T^\mu}_\nu = \mathrm{diag}(\rho, P, P, P), \tag{4.4}$$

where $\rho = \rho(t)$ is the total energy density and $P = P(t)$ the total pressure of the "cosmological fluid". Equating the components of the Einstein tensor to the relevant component of the energy-momentum tensor yields the Einstein equations in an FRW background:

$$H^2 = \frac{8\pi G}{3}\rho, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\big(\rho + 3P\big). \tag{4.5a}$$

The first of these equations is the Friedmann equation and the second is the Raychaudhuri equation. The final equation we must calculate in the FRW background is the conservation equation, $\nabla_\mu T^\mu{}_\nu = 0$, which yields

$$\dot{\rho} = -3H(\rho + P). \tag{4.5b}$$

The cosmological equations (4.5) are not yet closed: there is no rule for the evolution for the pressure P. The usual way to close the equations is via an equation of state,

$$P = w\rho, \tag{4.6}$$

which specifies the pressure of a substance in terms of its density. The equation of state of baryonic matter, radiation and a cosmological constant are respectively

$$w_{\rm m} = 0, \qquad w_{\rm r} = \frac{1}{3}, \qquad w_\Lambda = -1. \tag{4.7}$$

When the universe is dominated by different components, the evolution of the scale factor, $a(t)$, is different. To obtain the different behaviour, first the fluid equation (4.5b) is integrated to obtain the dependance of the energy density on scale factor when the universe is dominated by a component with a particular equation of state, yielding

$$\rho(a) \propto a^{-3(1+w)}. \tag{4.8}$$

We now use (4.8) in the Friedmann equation to provide an equation describing how the scale factor evolves when the universe is dominated by a component with equation of state w:

$$a(t) \propto t^{\frac{2}{3(1+w)}}. \tag{4.9}$$

The density and scale factor evolution for a universe dominated by matter, radiation and a cosmological constant is

$$w_{\rm m} = 0 \qquad \Rightarrow \qquad \rho_{\rm m} \propto a^{-3}, \qquad a \propto t^{2/3}, \tag{4.10a}$$

$$w_r = \frac{1}{3} \qquad \Rightarrow \qquad \rho_{\rm r} \propto a^{-4}, \qquad a \propto t^{1/2}, \tag{4.10b}$$

$$w_\Lambda = -1 \qquad \Rightarrow \qquad \rho_\Lambda \propto \text{const}, \qquad a \propto e^{Ht}. \tag{4.10c}$$

By defining the critical density of the universe, $\rho_{\rm c} \equiv 3H_0^2/8\pi G$ to be the density required to produce a flat universe, and the density fractions, $\Omega_{\rm X} \equiv \rho_{\rm X}/\rho_{\rm c}$, the Friedmann equation can be written as

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\rm r}}{a^4} + \frac{\Omega_{\rm m}}{a^3} + \Omega_\Lambda. \tag{4.11}$$

This can be used in conjunction for the equation of a light-ray (radial null geodesic), $\mathrm{d}t = a(t)\mathrm{d}r$, to obtain the distance r to an object at scale factor a,

$$r(a) = \frac{1}{H_0}\int_a^1 \frac{\mathrm{d}a}{\sqrt{\Omega_{\mathrm{r}} + a\Omega_{\mathrm{m}} + a^4\Omega_{\Lambda}}}. \tag{4.12}$$

By measuring the angle θ subtended by an object of physical length l, the angular diameter distance is given by

$$d_{\mathrm{A}}(a) = ar(a) = \frac{l}{\theta}. \tag{4.13}$$

So, by measuring θ for objects of known intrinsic lengths, one can obtain information about the cosmological parameters. It is common to use redshift $z \equiv 1/a - 1$.

B. Cosmological perturbations

The FRW metric (4.1) is a model for the geometry of the universe on the largest possible scales, where the content of the universe was taken to be a homogeneous and isotropic fluid. What this means, however, is that the metric is not capable of encoding information about the gravitational effects of localized structures in the universe. To be able to do this the gravitational field corresponding to this structure is assumed to be a small perturbation to the metric. There are many classic texts and reviews on cosmological perturbation theory in the literature, e.g. [2, 25–29].

The metric perturbed about a conformally flat FRW background is written as

$$\mathrm{d}s^2 = a^2(t)(\eta_{\mu\nu} + h_{\mu\nu})\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{4.14}$$

The perturbations $h_{\mu\nu}$ can be space and time dependent. We parameterize the $h_{\mu\nu}$ as

$$\mathrm{d}s^2 = a^2(\tau)\bigg[-(1+2\Phi)\mathrm{d}\tau^2 + 2N_i \mathrm{d}x^i \mathrm{d}\tau + (\delta_{ij} + h_{ij})\mathrm{d}x^i \mathrm{d}x^j \bigg], \tag{4.15}$$

where $\Phi = \Phi(\tau, \mathbf{x})$ is interpreted as the Newtonian gravitational potential, $N_i = N_i(\tau, \mathbf{x})$ as the lapse function and $h_{ij} = h_{ij}(\tau, \mathbf{x})$ as the spatial metric perturbations. Because the metric is perturbed, all geometrical quantities computed from the metric will also become perturbed from their value in an FRW background, as given in (3.17). In General Relativity, not all of the components of the metric (4.15) satisfy dynamical equations of motion and some of the components of the metric are merely constraints (this is nicely illustrated in the ADM formalism [30]). These correspond to gauge freedom in the theory, and so it is usual to make coordinate system choices to remove this freedom; there are four components which can be removed. Two of the popular choices are the *synchronous gauge* $\Phi = 0, N_i = 0$ and the *conformal Newtonian gauge* $N_i = 0, h_{ij} = -2\Psi\delta_{ij}$. In the conformal Newtonian gauge only scalar perturbations to the metric can be studied (the perturbations are in the form of two gravitational potentials, Φ, Ψ).

Armed with the parameterization of the perturbed metric (4.15) we must provide field equations governing the metric perturbations, and how the dynamics of the perturbations are determined by the content (which must also be perturbed). In General Relativity these governing equations are given by Einstein's field equations expanded to linear order in the perturbations to the metric,

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}. \tag{4.16}$$

The metric (4.15) is used to calculate the components of the perturbed Einstein tensor, δG^{μ}_{ν}; we are working in conformal coordinates so over-dots are derivatives with respect to conformal time and $\mathcal{H} \equiv \dot{a}/a$ is the conformal time Hubble parameter. In the synchronous gauge, the components of $\delta G^{\mu}{}_{\nu}$ are [31]

$$a^2 \delta G^0{}_0 = -\mathcal{H}\dot{h} + \frac{1}{2}\nabla^2 h - \frac{1}{2}\partial_i \partial_j h^{ij}, \tag{4.17a}$$

$$2a^2 \delta G^0{}_i = \partial_i \dot{h} - \partial_j \dot{h}^j{}_i, \tag{4.17b}$$

$$\begin{aligned} 2a^2 \delta G^i{}_j &= -\left[\ddot{h} + 2\mathcal{H}\dot{h} - \nabla^2 h + \partial_k \partial_l h^{kl}\right]\delta^i{}_j \\ &\quad + \ddot{h}^i{}_j + 2\mathcal{H}\dot{h}^i{}_j - \nabla^2 h^i{}_j + \partial^i \partial_k h^k{}_j + \partial_j \partial_k h^{ik} - \partial^i \partial_j h. \end{aligned} \tag{4.17c}$$

In the conformal Newtonian gauge the components of $\delta G^{\mu}{}_{\nu}$ are [32, 33]

$$a^2 \delta G^0{}_0 = 2\left[-\nabla^2 \Psi + 3\mathcal{H}\big(\dot{\Psi} + \mathcal{H}\Phi\big)\right], \tag{4.18a}$$

$$a^2 \delta G^0{}_i = -2\nabla_i \big(\dot{\Psi} + \mathcal{H}\Phi\big), \tag{4.18b}$$

$$a^2 \delta G^i{}_j = 2\left[\ddot{\Psi} + \big(\mathcal{H}^2 + 2\dot{\mathcal{H}}\big)\Phi + \big(\dot{\Phi} + 2\dot{\Psi}\big)\mathcal{H}\right]\delta^i{}_j + \left[\nabla^i \nabla_j - \delta^i{}_j \nabla^2\right]\big(\Psi - \Phi\big). \tag{4.18c}$$

It is convenient to write the components of the perturbed energy momentum tensor as

$$\delta T_{\mu\nu} = \delta\rho u_\mu u_\nu + \delta P \gamma_{\mu\nu} + 2(\rho + P) v_{(\mu} u_{\nu)} + P\Pi_{\mu\nu}, \tag{4.19}$$

where u_μ is a time-like unit vector $\gamma_{\mu\nu}$ a space-like tensor, v_μ a space-like vector and $\Pi_{\mu\nu}$ a spatial transverse-traceless tensor. These vectors and tensors satisfy

$$u^\mu u_\mu = -1, \quad v^\mu u_\mu = 0, \quad u^\mu \gamma_{\mu\nu} = 0, \quad \Pi^\mu{}_\mu = 0, \quad u^\mu \Pi_{\mu\nu} = 0. \tag{4.20}$$

The component $\delta\rho$ is interpreted as the perturbed density, δP as the perturbed pressure, v_μ as the velocity field and $\Pi_{\mu\nu}$ as the shear tensor. It is then usual to define the density contrast, $\delta \equiv \delta\rho/\rho$.

To be able to solve the perturbed gravitational field equations the equations of motion governing the perturbed sources, $\delta T^\mu{}_\nu$, must be provided. The relevant equation of motion is the perturbed conservation equation, $\delta(\nabla_\mu T^\mu{}_\nu) = 0$, which expands to yield

$$\nabla_\mu \delta T^\mu{}_\nu + \delta\Gamma^\mu{}_{\mu\alpha} T^\alpha{}_\nu - \delta\Gamma^\alpha{}_{\mu\nu} T^\mu{}_\alpha = 0. \tag{4.21}$$

Using (4.15) to parameterize the perturbed metric and (4.19) to parameterize the perturbed content, the time and space parts of the perturbed conservation equation respectively yield

$$\dot{\delta} = -(1+w)\left(\nabla_j v^j + \frac{1}{2}\dot{h}\right) - 3\mathcal{H}\left(\frac{\delta P}{\delta\rho} - w\right)\delta, \tag{4.22a}$$

$$\dot{v}_i \;=\; -\mathcal{H}\big(1-3w\big)v_i + \big(\partial_i\Phi - \mathcal{H}N_i\big) - \frac{1}{\rho(1+w)}\partial_i\delta P - \frac{w}{1+w}\nabla_j\Pi^j{}_i, \tag{4.22b}$$

where we have taken $\dot{w} = 0$ for simplicity. As noted for the background cosmological equations, the perturbed equations are not closed evolution equations until $\delta P, \Pi^i{}_j$ are specified by, for example, equations of state.

C. The dark side

By observing different cosmological and astrophysical systems (such as galaxy rotation curves, the cosmic microwave background, distances to supernovae, gravitational lensing and structure formation) and analyzing the data within the standard cosmological framework (GR to provide the gravitational theory and an FRW metric for the geometry), observers and theorists have come to the rather startling conclusion that about 96% of the universe is comprised of two forms of matter of which we have absolutely zero comprehension: *dark matter* and *dark energy* [34–36] (see also [37, 38]). This has rather profound implications for our understanding of a cosmological model. Dark matter was introduced to fix galaxy rotation curves and for structure formation, whilst dark energy was introduced as an explanation for the origin of the apparent observed acceleration of the universe. Within the framework of the standard GR + FRW cosmological paradigm, acceleration $\ddot{a} > 0$ only occurs whenever the energy density and pressure of the content satisfy $\rho + 3P < 0$, which requires the equation of state $w < -\frac{1}{3}$.

Dark matter has the same gravitating properties as Baryonic matter, in that it has negligible pressure, but it differs in that dark matter does not appear to interact with electromagnetic fields. Thus, our only hint as to the existence of dark matter comes from its gravitational effects.

Dark energy is rather more alien in nature, and apparently needs to be included as the currently dominant gravitating species in the content of the universe. The observation of apparent acceleration appears to suggest that dark energy must be a substance with equation of state $w_{\rm de} \approx -1$. No known substance has such an equation of state. In fact, the substance with this equation of state is highly exotic because it acts as some sort of "anti-gravity", gravitationally repelling objects.

The important point to realize is that when the observational data are analyzed through a specific gravitational and cosmological theory, 96% of the universe is required to be invented. This raises some serious and fundamental questions:

1. What is the dark energy and dark matter?

2. Is it appropriate to use the FRW metric on cosmological scales?

3. Is it appropriate to use GR on cosmological scales?

The way to "understand" these problems is to be clear about what it is that is assumed to be *a priori* true, and what it is that is deduced from observations given what is assumed to be true.

One possibility is that we could assume that GR is the appropriate gravitational theory and FRW is an accurate geometrical model for cosmological scales, in which case we must start to ask "what is the dark matter and the dark energy?". There are many theories in the literature, for example cosmological constant, quintessence, elastic dark energy, cold dark matter, warm dark matter and axions. The full list of theories is large and has been extensively studied in the literature (see, e.g. [39–43]). There are a few popular theories in the literature which provide a substance with an equation of state $w < -\frac{1}{3}$. The simplest is the cosmological constant, Λ, whose equation of state is $w_\Lambda = -1$; there is no variation in time or space of the energy density or equation of state of Λ. The simplest alternative to Λ is the dynamical minimally coupled homogeneous "quintessence" scalar field ϕ [44]. The Lagrangian density and energy-momentum tensor of quintessence is

$$\mathcal{L}_{(\phi)} = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi), \qquad T^{\mu\nu}_{(\phi)} = \nabla^\mu\phi\nabla^\nu\phi - g^{\mu\nu}\mathcal{L}, \tag{4.23}$$

where $V = V(\phi)$ is the potential function. Because the scalar field is homogeneous, we have $\phi = \phi(t)$ and the energy density, $\rho_{(\phi)}$, and pressure, $P_{(\phi)}$, become

$$\rho_{(\phi)} = \frac{1}{2}\dot{\phi}^2 + V, \qquad P_{(\phi)} = \frac{1}{2}\dot{\phi}^2 - V, \tag{4.24}$$

which allows the value of the equation of state parameter of the quintessence field to be computed,

$$w_\phi = \frac{P_{(\phi)}}{\rho_{(\phi)}} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} \tag{4.25}$$

If the content of the universe were to be dominated by the quintessence scalar field then the acceleration condition $\rho_\phi + 3P_\phi < 0$ becomes $V > \dot{\phi}^2$. That is, whenever the kinetic energy of the scalar field is less than its potential energy the quintessence field can drive an acceleration of the universe. A simple generalization of the quintessence field is k-essence [45], The Lagrangian density and energy-momentum tensor of k-essence is given by

$$\mathcal{L} = \mathcal{L}(\mathcal{X}, \phi), \qquad T_{\mu\nu} = \mathcal{L}_{,\mathcal{X}}\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\mathcal{L}, \tag{4.26}$$

where the kinetic term, $\mathcal{X} \equiv \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$. For a homogeneous scalar field, the energy density and pressure are given by

$$\rho_K = \mathcal{L}_{,\mathcal{X}}\dot{\phi}^2 - \mathcal{L}, \qquad P_K = -\mathcal{L}. \tag{4.27}$$

An alternative idea is to still assume that GR is appropriate, but question the validity of applying the FRW metric on cosmological scales, and begin to study inhomogeneous cosmologies [46–50]. We must understand whether or not the effect of localized matter distributions (such as galaxies, or clusters of galaxies) could produce something which we would interpret as being cosmological acceleration if we were ignorant of these matter distributions on cosmological scales. The fact remains that by using the FRW metric we are ignorant of the matter distributions. There are a growing number of studies in the literature on inhomogeneous universe (see, e.g. [51–56]), but this field is nowhere near as developed as the study of modified gravity or dark energy theories.

The third step is to question the validity of GR on cosmological scales. It is the purpose of the rest of this thesis to discuss and unpack this point.

V. MODIFIED GRAVITY

There is a surge in interest in gravity theories that are, in some way, different from General Relativity. These theories are collectively known as *modified gravities.* Studying modified gravities can be motivated in a few different ways. First of all, there is "pure academic interest": attempting to understand the structure of different types of field equations to build a full picture of how gravity works in the broadest sense. Secondly, the discovery of the dark side could be explained by the existence of some new gravitational theory on large scales. We will now simplistically elucidate on a historical precedent for this second point.

A few hundred years ago, humanity had almost a complete ignorance of how gravity worked, or how to determine the force a body felt due to a distribution of mass. Over time a series of earth based experiments began to be envisaged and constructed which enabled an empirical law to be written down which links the force due to to gravity, F, of two bodies separated by a distance r. This is of course the inverse square law: $F \propto 1/r^2$. Careful experimentation also enabled the constant of proportionality to be determined: $F = GMm/r^2$. The constant G is Newton's gravitational constant, M and m are the masses of the two bodies separated by a distance r, and F is the force of gravity between the bodies, pulling the bodies together. This equation encapsulates the essence of Newton's gravitational theory.

Once Newtonian gravity was formulated, more earth based experiments were constructed to test Newton's gravitational theory. When all the experiments returned the same results and agreed with the predictions from Newtonian gravity, experimenters and theorists grew in confidence that the theory is correct *in the physical scenarios in which the law was empirically deduced.* This confidence was then extended and Newtonian gravity was used to generate predictions for physical scenarios in which the law was not empirically deduced. For instance, the orbits of the planets in the Solar System or the behaviour of the paths of light next to the Sun.

Over time, theoretical predictions were extracted from Newtonian gravity and tested to high precision. As of the mid-1800's, the results can be crudely summarized as follows: the observed orbits of all planets *except two* agreed precisely with the predictions from Newtonian gravity. The closest planet to the Sun, Mercury, had a "wobble" in its orbit which could not be explained by applying Newton's gravitational theory to the Mercury-Sun system, and the orbit of Uranus had orbital discrepancies.

One of the explanations for these discrepancies was to invent two extra planets. Vulcan was predicted to exist close to the Sun, with the properties of Vulcan being chosen such that a Newtonian calculation of the Sun-Mercury-Vulcan system returned an answer which explained the observed wobble of Mercury. Neptune was predicted to exist in the vicinity of Uranus, whose properties were also chosen in such a way as to fix observed discrepancies.

Neptune was subsequently searched for and found, but Vulcan was not found (and indeed does not exist). The reason that Vulcan does not exist is actually because it was the expectation that Newton's understanding of gravity holds so close to the Sun which was false. Newtonian gravity was deduced on the earth which is a low-curvature environment, and fails in high-curvature environments such as those around the Sun. Einstein constructed his theory of General Relativity and used it to predict the dynamics of the Sun-Mercury system (where Newtonian gravity had failed). The GR prediction matched observation spectacularly well (and still provided the same predictions for the other planets as Newtonian gravity), without requiring Vulcan to be invented. The key was that Newtonian gravity was not designed to be applied in extreme gravitational fields, whereas GR was.

When we assume that the universe is comprised of matter we understand (photons, baryons, neutrinos etc), and that the cosmological principle holds, the predictions of GR and observations are completely incompatible. Only when we *impose* that dark matter and dark energy is the dominant contribution to the matter content do the predictions and experimental observations become compatible.

This story provides a powerful philosophical point: applying the Newtonian gravitational theory to a system it was never tested in had the consequence of requiring an entire planet to be invented. Today, applying GR to cosmological scales has meant that 96% of the universe must be invented. The analogy should now be obvious: perhaps GR is not the gravitational theory that should be applied on cosmological scales. Instead, perhaps some new "modified" gravity theory should.

Applying the framework of General Relativity to a system as massive as a galaxy, or indeed to the whole universe, is entirely an extrapolation of the assumption of the validity of GR as *the* gravitational theory. The discrepancy between predictions from the GR and the "standard matter" theoretical model and experimental observations has motivated a search for new gravitational theories. Preferably, this new theory will produce predictions about our universe which are compatible with experimental observations without the need for introducing dark energy (or at least, dark energy will not need to be the dominant matter contribution). However, it is necessary that this new gravitational theory has Newtonian and GR limits so that in the regimes where we know GR holds, the new gravitational theory satisfies current constraints from e.g. tests of the equivalence principle via torsion balances [57] or lunar laser ranging [58, 59], frame dragging around the earth [60] or in the strong field limit around binary pulsar systems [61].

It is worth recapping that GR provides a specific set of equations for determining the components of the metric, $g_{\mu\nu}$, from a matter content, through Einstein's field equations,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{5.1}$$

This field equation can be derived from the Einstein-Hilbert action,

$$S_{\mathrm{GR}} = \int \mathrm{d}^4x\,\sqrt{-g}\,\mathcal{L}_{\mathrm{EH}} + S_{\mathrm{m}}[g_{\mu\nu},\chi], \qquad \mathcal{L}_{\mathrm{EH}} = R, \tag{5.2}$$

where R is the Ricci scalar and S_{m} is the action of the matter fields χ. The popular way to construct a modified gravity theory is to provide extra terms or extra fields to the gravitational Lagrangian density. In general we will use an action

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\,\mathcal{L}, \tag{5.3}$$

where $\mathcal{L}$ is the gravitational Lagrangian density, and contains geometrical terms (such as the Ricci scalar), any extra gravitational mediators and all matter fields (which are usually collected into a matter Lagrangian $\mathcal{L}_{\mathrm{m}}$). Modified gravity theories provide alternative field equations, meaning that the metric will respond differently to the same matter content. As remarked in section III D, a well posed variational principle requires some specification of the behaviour of the theory on the boundary (e.g. the Gibbons-Hawking term in GR). In the literature it is usually "assumed" that such a boundary term exists; only very recently [62] were the explicit forms of the boundary conditions worked out for general scalar-tensor theories.

A. A catalogue of modified gravity theories

There are a large number of modified gravity theories in the literature but here we will provide a brief overview of some popular modified gravity theories; for a recent extensive review see [63]. Most modified gravity theories can be classified broadly into two categories: (a) only the metric is used to mediate gravity and (b) extra mediators of gravity are introduced; this classification is not entirely unambiguous (for example, metric-only theories can be rewritten as scalar-tensor theories), but it is useful for our purposes of providing a brief review.

1. Tensor theories

To begin our brief review of modified gravity theories, we will consider theories which only use the metric to mediate gravity: these are called *tensor theories*. The Lagrangian of a tensor theory is of the general form

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha \partial_\beta g_{\mu\nu}, \ldots). \tag{5.4}$$

The simplest tensor theory is the cosmological constant [64–66], whereby the gravitational Lagrangian density is

$$\mathcal{L} = R - 16\pi G \mathcal{L}_{\rm m} - \Lambda, \tag{5.5}$$

whose gravitational field equations are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{5.6}$$

A simple way to extend the gravitational Lagrangian is the manner in which the Ricci scalar appears (in the Einstein-Hilbert action the Ricci scalar appears linearly). These are called $f(R)$ theories [67–69]. The gravitational action is modified to include an arbitrary function of the Ricci scalar:

$$\mathcal{L} = R + f(R) - 16\pi G \mathcal{L}_{\rm m}, \tag{5.7}$$

whose gravitational field equations are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \big(R_{\mu\nu} + g_{\mu\nu}\nabla^\alpha\nabla_\alpha - \nabla_\mu\nabla_\nu\big) f' - \frac{1}{2} f g_{\mu\nu}, \tag{5.8}$$

where $f' \equiv {\rm d}f/{\rm d}R$. One can then begin to envisage a generalization of such gravitational theories, for example where arbitrary functions of curvature invariants appear,

$$\mathcal{L} = R + f(R, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}) - 16\pi G \mathcal{L}_{\rm m}. \tag{5.9}$$

A further example of these types of theories are the Gauss-Bonnet gravities [70–73], where the gravitational Lagrangian contains the Gauss-Bonnet term,

$$\mathcal{L} = \mathcal{L}(R, \mathcal{G}), \qquad \mathcal{G} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}. \tag{5.10}$$

2. Scalar-tensor theories

The next type of modified gravity theory we consider contains extra mediators of gravity. The simplest such "extra mediator" is a scalar field, ϕ, so that we talk of scalar-tensor gravity theories. The Lagrangian of a scalar-tensor theory will be of the form

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha \partial_\beta g_{\mu\nu}, \ldots, \phi, \partial_\mu \phi, \ldots). \tag{5.11}$$

A simple example of theories of this class uses a function of a dynamical scalar field to couple to the Ricci scalar in the Lagrangian,

$$\mathcal{L} = \mathfrak{a}(\phi)R - \mathfrak{b}(\phi)\nabla^\mu\phi\nabla_\mu\phi - 2V(\phi), \tag{5.12}$$

where $\mathfrak{a}, \mathfrak{b}, V$ are arbitrary functions of the scalar field (V is the potential term). The field equations for the metric scalar field are respectively given by

$$\mathfrak{a}G_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\mathfrak{a} = 8\pi G T_{\mu\nu} + T^{(\phi)}_{\mu\nu}, \tag{5.13a}$$

$$\mathfrak{b}\Box\phi = V' - \frac{1}{2}\mathfrak{a}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{2}\mathfrak{a}'R, \tag{5.13b}$$

where for convenience we have defined

$$T^{(\phi)}_{\mu\nu} \equiv \mathfrak{b}\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\alpha\phi\nabla_\alpha\phi - 2\frac{V}{\mathfrak{b}}\right)\right]. \tag{5.14}$$

Brans-Dicke theory [74] is a specific example, where the functions $\mathfrak{a}, \mathfrak{b}$ are taken to be

$$\mathfrak{a}(\phi) = \phi, \qquad \mathfrak{b}(\phi) = \frac{\omega(\phi)}{\phi}. \tag{5.15}$$

There has been a recent surge in interest of theories whose Lagrangians contain two derivatives of a scalar field, but whose field equations are at most of second order (naively, a Lagrangian with two derivatives has fourth order field equations). An example is the kinetic gravity braiding theory [75–77],

$$\mathcal{L}_{\text{KGB}} = K(\phi, \mathcal{X}) + G(\phi, \mathcal{X})\Box\phi, \tag{5.16}$$

where $\Box\phi \equiv \nabla^\mu\nabla_\mu\phi$. The theory is called "braided" because the Ricci tensor explicitly enters into the equation of motion for the scalar field in the process of making the field equations second order. Horndeski's theory [78] is constructed to be the most general scalar-tensor theory in 4D with second order field equations, and has recently been rediscovered and studied by a number of authors [63, 79–82]. The covariant galileon theory [63, 83–86], is another example of a theory whose Lagrangian contains the metric, curvature tensors, first and second derivatives of the scalar field. The theory is constructed so that the field equations contain at most second order in derivatives of the metric (the galileon and Horndeski theories have been shown to be equivalent to each other [87]). The covariant galileon Lagrangian is given by

$$\mathcal{L} = \sum_i c_i \mathcal{L}_i, \tag{5.17}$$

where c_i are dimensionless constants and the Lagrangians $\mathcal{L}_i$ are purely "kinetic" functions of the galileon scalar field π, given by

$$\mathcal{L}_1 = \frac{1}{2}\nabla_\mu\pi\nabla^\mu\pi, \qquad \mathcal{L}_2 = \frac{1}{2M^3}\nabla_\mu\pi\nabla^\mu\pi\Box\pi, \tag{5.18a}$$

$$\mathcal{L}_3 = \frac{1}{2M^6}\nabla_\mu\pi\nabla^\mu\pi\left[2(\Box\pi)^2 - 2\nabla^\mu\nabla^\nu\pi\nabla_\mu\nabla_\nu\pi - \frac{1}{2}R\nabla_\mu\pi\nabla^\mu\pi\right], \tag{5.18b}$$

$$\begin{aligned}\mathcal{L}_4 \;=\; & \frac{1}{2M^9}\nabla_\mu\pi\nabla^\mu\pi\Big[(\Box\pi)^3 - 3(\Box\pi)\nabla^\mu\nabla^\nu\pi\nabla_\mu\nabla_\nu\pi \\ & +2\nabla^\nu\nabla_\mu\pi\nabla^\rho\nabla_\nu\pi\nabla^\mu\nabla_\rho\pi - 6\nabla_\mu\pi\nabla^\mu\nabla^\nu\pi\nabla^\rho\pi G_{\nu\rho}\Big],\end{aligned} \tag{5.18c}$$

where $R, G_{\mu\nu}$ are the Ricci scalar and Einstein tensor respectively. The field equations of this theory can be found in, e.g., [84]. Recently, highly stringent constraints were placed upon the galileon theory [86, 88] appearing to suggest that the theory is strongly disfavored by cosmological data sets.

The tensor theories we discussed in section V A 1 can be thought of as being non-minimal scalar-tensor theories. To see this, consider for a moment a scalar-tensor theory given by

$$S = \int \mathrm{d}^4\sqrt{-g}\,\Big[R + f(\psi) + f'(\psi)(R-\psi)\Big], \tag{5.19}$$

where we note that the Ricci scalar appears linearly and coupled to a non-dynamical scalar field $\phi \equiv f'(\psi)$ where the scalar field ψ has some potential term, $f(\psi)$. By extremizing the variation $\delta S = 0$, one obtains the constraint $f''(\psi) = 0$ or $R = \psi$. Assuming that $f(\psi)$ is chosen such that $f''(\psi) \neq 0$, then substituting $R = \psi$ into the action (5.19) yields

$$S = \int \mathrm{d}^4x\sqrt{-g}\,\Big[R + f(R)\Big], \tag{5.20}$$

which is indeed the action for an $F(R)$ theory. This idea can be generalized [89] for theories containing other curvature tensors. Let us write the scalar-tensor theory

$$\begin{aligned}S \;=\; & \int \mathrm{d}^4x\sqrt{-g}\,\Big[R + f(\phi_1,\phi_2,\phi_3) + f_1(R-\phi_1) \\ & +f_2(R^{\mu\nu}R_{\mu\nu} - \phi_2) + f_3(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - \phi_3)\Big],\end{aligned} \tag{5.21}$$

where $f_i \equiv \frac{\partial f}{\partial\phi_i}$. Then, if $\det\partial^2 f/\partial\phi_i\phi_j \neq 0$, the equations of motion of the ϕ_i yield a set of conditions $\phi_1 = R, \phi_2 = R^{\mu\nu}R_{\mu\nu}, \phi_3 = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, so that the action (5.21) is equivalent to

$$S = \int \mathrm{d}^4x\sqrt{-g}\,\Big[R + f(R, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta})\Big], \tag{5.22}$$

which is the action for "just" a tensor theory.

3. Tensor-vector theories

One can also introduce vector fields into the mediating sector of gravity. The Lagrangian of a general vector-tensor theory will be of the form

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha \partial_\beta g_{\mu\nu}, \ldots, A^\mu, \partial_\nu A^\mu, \ldots). \tag{5.23}$$

The most popular example of these classes of theories is the Einstein-æther theory [90–94], where a vector field A^μ is introduced into the gravitational Lagrangian and whose length is constrained to unity, with a generalized Maxwell kinetic term. The Lagrangian density for the linear Einstein-æther theory is

$$S = -\frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} \Big[R + K^{\mu\nu\alpha\beta} \nabla_\mu A_\alpha \nabla_\nu A_\beta + \lambda (A^\mu A_\mu - 1) \Big], \tag{5.24}$$

where

$$K^{\mu\nu\alpha\beta} = c_1 g^{\mu\nu} g^{\alpha\beta} + c_2 g^{\mu\alpha} g^{\nu\beta} + c_3 g^{\mu\beta} g^{\nu\alpha} + c_4 A^\mu A^\nu g^{\alpha\beta} \tag{5.25}$$

represents a generalization of the Maxwell kinetic term. The $\{c_i\}$ are constants and λ is a Lagrange multiplier whose role is to enforce the unit time-like constraint.

4. Tensor-vector-scalar theories

A gravitational theory containing scalar, vector and tensor fields can be written as

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \ldots, \phi, \partial_\mu \phi, \ldots, A^\mu, \partial_\nu A^\mu, \ldots). \tag{5.26}$$

By far the most popular modified gravity theory containing tensor, vector and scalar fields is TeVeS [95–100]. In the simplest version of TeVeS the action is given by

$$S \;=\; 16\pi G S_{\mathrm{m}}[g_{\mu\nu}, \chi, \nabla_\mu \chi] + \int \mathrm{d}^4 x \sqrt{-\tilde{g}} \Big[\mathcal{L}_{\mathrm{g}} + \mathcal{L}_{\mathrm{A}} + \mathcal{L}_\phi \Big]. \tag{5.27}$$

There are two metrics: the matter sector only couples to $g_{\mu\nu}$, and the gravitational sector is constructed from $\tilde{g}_{\mu\nu}$, which is related to the matter metric via a "disformal" transformation,

$$g_{\mu\nu} = e^{-2\phi} (\tilde{g}_{\mu\nu} + A_\mu A_\nu) - e^{2\phi} A_\mu A_\nu. \tag{5.28}$$

The Lagrangians that construct the gravitational sector are given by

$$\mathcal{L}_{\mathrm{g}} = \tilde{R}, \tag{5.29a}$$

$$\mathcal{L}_{\mathrm{A}} = -\frac{1}{2} k F^{\mu\nu} F_{\mu\nu} + \lambda (A^\mu A_\mu + 1), \tag{5.29b}$$

$$\mathcal{L}_\phi = -\mu (\tilde{g}^{\mu\nu} - A^\mu A^\nu) \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - V(\mu) \tag{5.29c}$$

TeVeS has been shown [101, 102] to be equivalent to a vector-tensor theory where the vector field is not of unit norm (which is the case in æther theories).

B. Einstein & Jordan frames

It is possible to find an equivalence between a set of theories where matter is universally coupled to the metric but the Ricci scalar is explicitly coupled to an "extra" scalar field (i.e. the Lagrangian contains a term of the form ϕR, so called scalar-tensor theories) and a set of theories where the scalar field and Ricci scalar are minimally coupled (i.e. there are no longer any terms of the form ϕR), but matter couples to a different metric to that generated by the gravitational field equations. The frame in which matter universally couples to the gravitating metric is called the *Jordan frame* and the frame in which matter couples to a different metric is called the *Einstein frame*. The actions in these two frames take on the schematic form

$$S_{\text{Jordan}} = \int \mathrm{d}^4x\sqrt{-g}\,\phi R + S_{\text{m}}[g_{\mu\nu};\chi], \tag{5.30}$$

$$S_{\text{Einstein}} = \int \mathrm{d}^4x\sqrt{-g}\left[R + \frac{1}{2}(\nabla\phi)^2\right] + S_{\text{m}}[A^2(\phi)g_{\mu\nu};\chi]. \tag{5.31}$$

Clearly, performing calculations with the Einstein frame action (5.31) will be much simpler than calculations with the Jordan frame action (5.30) because the Ricci scalar appears only linearly and uncoupled to any extra fields in the Einstein frame action. However, theories usually "present themselves" in Jordan frame form (for example, $F(R)$ theories can be shown to be equivalent to a theory in the Jordan frame with non-minimal coupling between the Ricci scalar and a scalar field). So, we would like to obtain an understanding as to how to transform between the Jordan and Einstein frames. The key to understanding lies through conformal transformation of the metric.

This gives rise to the chameleon mechanism [103–105]. The coupling term causes the bare potential that the scalar field feels to be shifted, $V_{\text{eff}}(\phi) = V_{\text{bare}}(\phi) + \rho A(\phi)$, where ρ is the energy density of a non-relativistic source. This causes a screening of the modification of the gravity theory in the vicinity of the source: what this means is that experiments which are designed to test deviations from GR in the vicinity of a planet, for example, will still detect GR.

We will show how to transfer between the Jordan (5.30) and Einstein (5.31) frames. We will start our calculation by stating some useful identities for conformal transformation (see e.g. [1, 19, 106]). Performing a conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ in n-dimensions, the measure and Ricci scalar transform as

$$\sqrt{-g} \to \sqrt{-\tilde{g}} = \sqrt{-g}\,\Omega^n, \tag{5.32}$$

$$R \to \tilde{R} = \Omega^{-2}\left[R - 2(n-1)\frac{\Box\Omega}{\Omega} - (n-1)(n-4)\frac{1}{\Omega^2}\nabla^\mu\Omega\nabla_\mu\Omega\right] \tag{5.33}$$

Specifically, in $n = 4$-dimensions,

$$\sqrt{-\tilde{g}} = \sqrt{-g}\,\Omega^4, \qquad \tilde{R} = \Omega^{-2}\left[R - 6\Omega^{-1}\Box\Omega\right]. \tag{5.34}$$

We now use these relations to perform a conformal transformation upon a theory containing a coupling between a scalar field ϕ and the Ricci scalar R in $n = 4$,

$$\int \mathrm{d}^4x\sqrt{-g}\,\phi R \to \int \mathrm{d}^4x\sqrt{-\tilde{g}}\,\tilde{\phi}\tilde{R} = \int \mathrm{d}^4x\sqrt{-g}\,\phi\Omega^2\Big(R - 6\Omega^{-1}\Box\Omega\Big), \tag{5.35}$$

and if we choose the conformal factor Ω to be related to the scalar field ϕ such that $\Omega^2\phi = 1$ then

$$\begin{aligned}\int \mathrm{d}^4x\sqrt{-g}\,\phi R &\to \int \mathrm{d}^4x\sqrt{-g}\,\Big(R - 6\Omega^{-1}\Box\Omega\Big)\\ &= \int \mathrm{d}^4x\sqrt{-g}\,\Big(R - \frac{6}{\Omega^2}\nabla^\mu\Omega\nabla_\mu\Omega - 6\nabla^\mu(\Omega^{-1}\nabla_\mu\Omega)\Big).\end{aligned} \tag{5.36}$$

The final term is a total derivative and is usually ignored (it is only a surface term), leaving

$$\int \mathrm{d}^4x\sqrt{-g}\,\phi R \;\to\; \int \mathrm{d}^4x\sqrt{-g}\,\Big(R - \frac{6}{\Omega^2}\nabla^\mu\Omega\nabla_\mu\Omega\Big). \tag{5.37}$$

If we now write the scalar field as $\Omega = e^{\frac{1}{\sqrt{12}}\psi}$ (i.e. $\psi = -\sqrt{3}\ln\phi$) then

$$\int \mathrm{d}^4x\sqrt{-g}\,\phi R \;\to\; \int \mathrm{d}^4x\sqrt{-g}\,\Big(R - \frac{1}{2}\nabla^\mu\psi\nabla_\mu\psi\Big). \tag{5.38}$$

What we have done, therefore, is show that a theory whose Lagrangian contains an explicit coupling between a scalar field ϕ and the Ricci scalar is equivalent to a theory without that coupling, but with the introduction of a canonical kinetic term for the scalar field. That is, we have transformed from a scalar-tensor theory with explicit coupling to a minimally coupled scalar-tensor theory.

VI. SUMMARY

The current status of gravitational physics can summarized as follows: the spacetime arena appears to be influenced by the gravitating content of the universe, and GR provides a specific set of field equations for how the spacetime is determined for a given content. However, observational data are completely incompatible with the predictions from GR with the metric of a homogeneous isotropic universe, unless dark energy is introduced. Modifying the gravitational physics away from GR is one way out of this situation, although one of the major obstacles that modified gravity theories face is that all theories must posses a Newtonian limit compatible with all Solar System tests to pass current observational constraints.

It is fair to say, therefore, that we do not yet have a complete theoretical picture that is consistent with the data and which answers the simple question "*how does gravity work?*"

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