Solitons

Jonathan A. Pearson*
Jodrell Bank Centre for Astrophysics,
The University of Manchester,
Manchester M13 9PL, U.K
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These are notes for a talk on solitons. It is basically mathematical physics. We are not mathematically rigorous. We intend to teach an intuitive understanding of solitons, what they are and a specific example in detail. We will continue with a qualitative description of Q-balls.

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Contents

I. What is a soliton? 1

II. Reprise of field theory 1
A. Potentials 2

III. Non-localized configurations 2

IV. Examples 2
A. Domain walls 3
B. Q-balls 4
C. Vortons 5

References 5

I. WHAT IS A SOLITON?

A soliton is a localized configuration of energy. Construct from continuous fields. Note: not like "square well" objects: everything is smooth and continuous. Not "points". There is no universal definition of a soliton (noone agrees on a single definition).

Use to describe objects in real life.

All objects in reality are smooth continuous objects. If these objects are to be generated (or, understood) using fields, then the fields must be continuous and smooth. The "shape" of a field is a consequence of the field being a solution to a differential equation: the differential equation is constructed from physical premises, as are the boundary conditions.

Simple examples

- Bore on river severn - a solitary wave
- Atomic nuclei
- Whirlpools - what is the structre of the "mouth" of the whirlpool? Why is it in one place and not everywhere?
- Topological defects (cosmic strings – monopoles – domain walls). Very important in cosmology, condensed matter physics & liquid Helium.
- The entire Universe - the Universe can be thought about as a domain wall.

II. REPRISE OF FIELD THEORY

The study of solitons relies heavily on field theory. Here we introduce our notation and recap some important concepts.

Notation:

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \quad (1)$$

so that

$$\partial_0 = \partial_t = \frac{\partial}{\partial t}, \qquad \partial_2 = \partial_y = \frac{\partial}{\partial y}. \quad (2)$$

Also, the covariant Laplacian (D'albertain)

$$\partial_\mu \partial^\mu \phi = \Box\phi = \ddot{\phi} - \nabla^2\phi. \quad (3)$$

For those that care, $\partial_\mu\partial^\mu = \eta_{\mu\nu}\partial^\mu\partial^\nu$, where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric in flat spacetime (thats where the minus sign comes from!).

What is a field? A field is a continuous machine that takes the values of the coordinates of a point in spacetime and returns a number/vector:

$$\phi : S \to \mathbb{R}^N. \quad (4)$$

Usually write this as $\phi = \phi(x^\mu) = \phi(t, \mathbf{x})$. The next job is to figure out what defines the value of the field at each location in spacetime. This comes from solving the (equations of motion, boundary conditions). The equations of motion come from the fundamental piece of a field theory: the Lagrangian density.

Lagrangian density $\mathcal{L} = T - V$. Lagrangian density .vs. Lagrangian:

$$L = \int \mathrm{d}^3x\, \mathcal{L}, \quad (5)$$

*Electronic address: jp@jb.man.ac.uk

so that action:

$$S=\int \mathrm{d}tL=\int \mathrm{d}t\mathrm{d}^3x\,\mathcal{L} \tag{6}$$

The Lagrangian density contains all information (actually, the action does: in curved spacetimes the action introduces some terms in the metric). From $\mathcal{L}$ one can derive (i) the equations of motion (ii) energy (iii) pressure (iv) energy density (v) its gravitational field (vi) momentum (vii) conserved quantities.

Euler-Lagrange equations Action minimizing paths/fields. Field generalization of ones you know about from second year

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right)-\frac{\partial \mathcal{L}}{\partial \phi}=0. \tag{7}$$

Energy momentum tensor

$$T^\mu_\nu=\partial^\mu\phi\partial_\nu\phi-\delta^\mu_\nu\mathcal{L} \tag{8}$$

Energy density $\mathcal{E}=T^0_0$. Pressure $p=-\frac{1}{3}T^i_i$ (the trace). Momentum $P_i=T^0_i$. Angular momentum $P_\theta=T^0_\theta$.

Example Take

$$\mathcal{L}=\frac{1}{2}\dot{\phi}^2-V(\phi), \tag{9}$$

Euler-Lagrange equation:

$$\ddot{\phi}=-V', \tag{10}$$

which is $F=ma$. Energy density:

$$\mathcal{E}=\dot{\phi}^2+V, \tag{11}$$

which is of the form: $T+V$.

Example Take

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi) \tag{12}$$

$$= \frac{1}{2}\dot{\phi}^2-\frac{1}{2}|\nabla\phi|^2-V(\phi). \tag{13}$$

For those that worry about it: ϕ is a global real scalar field. Can formulate this for vector fields, complex fields and for gauge invariant fields. Gauge invariance requires inclusion of photon fields. Hence, equation of motion:

$$\partial_\mu\partial^\mu\phi=\ddot{\phi}-\nabla^2\phi = -\frac{\mathrm{d}V}{\mathrm{d}\phi} \tag{14}$$

Energy density

$$T^0_0=\mathcal{E}=\dot{\phi}^2+|\nabla\phi|^2+V, \tag{15}$$

energy

$$E=\int \mathrm{d}^3x\,\mathcal{E}. \tag{16}$$

Note that spatial gradients contribute to energy. So do time varying fields. As does the potential.

A. Potentials

Fields evolve to minimize their potential, so that the energy is minimized. e.g.

$$V(\phi)=\frac{1}{2}m\phi^2, \tag{17}$$

so that

$$\phi_{\mathrm{min}}=0. \tag{18}$$

This potential has a parabolic form, and is symmetric under reflection about its minimum.

Now consider

$$V(\phi)=\frac{\lambda}{4}\left(\phi^2-\eta^2\right)^2, \tag{19}$$

so that

$$\phi_{\mathrm{min}}=\pm\eta. \tag{20}$$

This potential is of Mexican-hat form, and is not invariant under reflection about its minimum (a very important concept we will come back to).

III. NON-LOCALIZED CONFIGURATIONS

Lagrangian density

$$\mathcal{L}=\partial_\mu\phi\partial^\mu\phi \tag{21}$$

(Euler-Lagrange equations) Wave equation

$$\frac{\partial^2\phi}{\partial t^2}=\frac{\partial^2\phi}{\partial x^2}, \tag{22}$$

solution

$$\phi(x,t)=Ae^{\mathrm{i}(\omega t-kx)}. \tag{23}$$

Note, $V=0$. This oscillation is infinite in extent. Energy density:

$$\mathcal{E}=\left(\omega^2+k^2\right)A^2, \tag{24}$$

this is a constant: plane waves have constant energy density: non-localized configurations.

IV. EXAMPLES

The job of constructing solitons can be boiled down to choosing a few things:

- a sensible potential,
- a sensible ansatz for your fields,
- the boundary conditions.

Some real field theory examples that can be constructed:
Q-balls - used in *Sunshine* to stop sun from working
Cosmic string
Superconducting string
Kinky vortons
Vortons
Skyrmions
Monopoles
Some of these have only been discovered in the past couple/3 years! Still a hot topic of research.

A. Domain walls

Construct a genuine theory with a soliton. Look at its phenomenology. Domain walls: my personal area of research (MPhys project & a few papers). Consider easiest possible example: useful because exact analytic solution is known to exist.

How to do it:

Write Mexican hat potential for ϕ real scalar field

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2 . \tag{25}$$

Has minima at $\phi = \pm\eta$. This is called the vacuum: the ϕ-field has two different vacuum values.

Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2 . \tag{26}$$

Equations of motion:

$$\ddot{\phi} - \nabla^2\phi = -\lambda\phi\left(\phi^2 - \eta^2\right) . \tag{27}$$

WANT a ***static*** solution in 1D with non-zero localized energy. Can do it by fixing the value of the field ϕ at $x = \pm\infty$ to be in different vacua (the $\phi = \pm\eta$ at $x = \pm\infty$). These boundary conditions define a topology.

Equation of motion assuming 1D and static is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \lambda\phi\left(\phi^2 - \eta^2\right) . \tag{28}$$

The solution is

$$\phi(x) = \eta \tanh\left(x\sqrt{\frac{\lambda\eta^2}{2}}\right) . \tag{29}$$

(verify by substitution, if you are so inclined!) The energy density is

$$\mathcal{E}(x) = V(\phi(x)) + \eta\sqrt{\frac{\lambda\eta^2}{2}}\,\mathrm{sech}^2\left(x\sqrt{\frac{\lambda\eta^2}{2}}\right) . \tag{30}$$

See Figure 1 and Figure 2 for plots of the solution $\phi(x)$ and energy density $\mathcal{E}(x)$.

These things are called *domain walls*: the field ϕ interpolates *smoothly* between $\phi = +\eta$ at $x = +\infty$ to $\phi = -\eta$ at $x = -\infty$. The "interpolating bit" at $x = 0$ is the domain wall: it separates regions of space where ϕ is in different minima. It is a soliton because it is in a localized place.

Notice that the choice of boundary conditions ***ensures*** that the soliton exists (this is an example of a topological soliton). Because the field is "anchored" at values which minimize the potential and these values are different, the field must interpolate (smoothly).

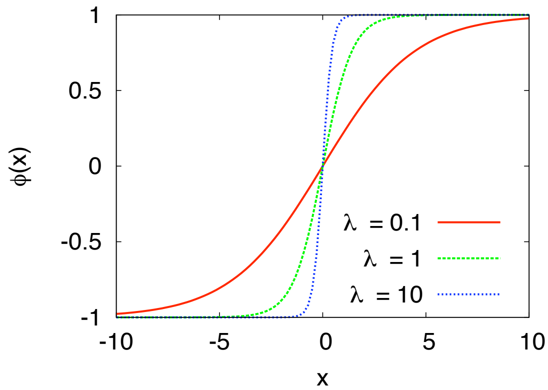

Figure 1: Domain wall solutions for various values of λ, fixing $\eta = 1$. Note that λ controls the steepness of the domain wall.

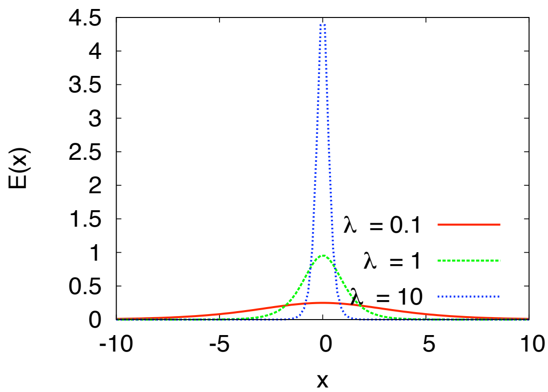

Figure 2: Domain wall energy densities for various values of λ, fixing $\eta = 1$. Note that λ controls the level of localization of the domain wall – this could have been guessed from Figure 1.

Note that changing λ changes the steepness of the curve. As $\lambda \to 0$ the curve becomes more shallow. Also, as $\lambda \to \infty$ the energy density becomes more localized.

If have time, look at videos of domain wall networks.

Domain walls are created in physics by a phase transition: steam-water-ice. Bar magnets.

A domain wall is a surface defect: the dimension of the domain wall is always one less than the dimension of the space it sits in. So, for example, the domain wall above was a point, 0-dimensional, sitting on the 1D line. In a 2D plane-space a domain wall is a line. Interestingly: in a 4D space a domain wall is a 3D volume – this idea has spurred the thought that the Universe is a domain wall.

With regards to the generation of domain walls from physical processes, consider the potential, which can be expanded out to give

$$V(\phi) = \frac{\lambda}{4}\phi^4 - \frac{\lambda}{2}\phi^2 + \frac{\lambda}{4}\eta^4. \tag{31}$$

Let us take a step back and consider a function

$$U(\gamma) = A\gamma^4 + B\gamma^2 + U_0. \tag{32}$$

The form of the minima of this potential depends on the sign of B. If $B > 0$ the potential has a single minimum (at the origin; it looks like a parabola). If $B < 0$ the single minimum breaks apart into two minima (i.e. becomes of a Mexican hat form). Notice that in the $B > 0$ regime the potential is symmetric about the minimum. For $B < 0$ the symmetry under reflection about the minimum is lost – this is called a symmetry breaking.

If one can dream up a physical process for B changing sign from being positive to negative, then one has dreamed up a physical mechanism for the production of a domain wall forming theory (the theory described above is written down in isolation from the rest of the universe). A nice example is making B a function of temperature. A physically motivated function is

$$B(T) = T - T_{\rm c}, \tag{33}$$

where $T_{\rm c}$ is a critical temperature. Then, for high temperatures $T > T_{\rm c}$, we have that $B > 0$ and the potential has a single minimum (no domain walls). For low temperatures $T < T_{\rm c}$ and therefore $B < 0$ and domain walls are allowed. This is the physical mechanism for producing domain walls in reality. Anything which decreases its temperature goes through a phase transition (for example, $T_{\rm c}$ for water is at its freezing point). Also recall that for low temperatures there is some symmetry which is broken. This can be exactly extended to much more complicated theories – such as an electroweak or strong symmetry breaking phase transition. The Universe went through a series of these phase transitions as it cooled down – and a consequence of these phase transitions is that we have electromagnetism and the strong nuclear force.

B. Q-balls

A Q-ball is a (spherically symmetric) ball of charge. Let us construct from a field $\sigma = \sigma(t, \mathbf{x})$. Will look at field theory of charge, and some qualitative solutions. There are lots of theories which produce Q-balls (i.e. lots of choices of the potential). Some theories even have analytic solutions.

Charge defined via

$$Q = \frac{1}{2{\rm i}}\int {\rm d}^3x\,(\dot{\sigma}\bar{\sigma} - \sigma\dot{\bar{\sigma}})\,. \tag{34}$$

This comes from Noether's theorem [1]. For charge to be present, something must be moving, $\dot{\sigma} \neq 0$, so non-static.

Just like domain wall case, can find stationary solution (note: difference between stationary and static). Thus, need to eliminate $\dot{\sigma}$-term, whilst leaving a charge Q behind.

If we write

$$\sigma(t, \mathbf{x}) = \psi(\mathbf{x})e^{{\rm i}\omega t} \tag{35}$$

then

$$Q = \omega\int {\rm d}^3x\,|\psi|^2. \tag{36}$$

Note that charge density

$$\rho_{\rm Q} = \omega|\psi|^2. \tag{37}$$

Hence, if $|\psi|$ in a region, there is zero charge in that region. Let us define

$$\Sigma \equiv \int {\rm d}^3x\,|\psi|^2, \tag{38}$$

then

$$Q = \omega\Sigma. \tag{39}$$

Notice that one cant think about Σ as being the volume of the blob or of the particle number [2].

Write a Lagrangian density:

$$\begin{aligned} \mathcal{L} &= \partial_\mu\sigma\partial^\mu\bar{\sigma} - V(\sigma) && (40)\\ &= |\dot{\sigma}|^2 - |\nabla\sigma|^2 - V(\sigma) && (41)\\ &= \omega^2|\psi|^2 - |\nabla\psi|^2 - V(\psi). && (42) \end{aligned}$$

Thus, the ω-term looks like a modification to the potential:

$$\mathcal{L} = -|\nabla\psi|^2 - U(\psi), \tag{43}$$

where the effective potential is

$$U(\psi) = V(\psi) - \omega^2|\psi|^2. \tag{44}$$

(infact, ω modifies the mass of the ψ-field) Hence, the Lagrangian density is now written in terms of a stationary field ψ, but has a charge.

Energy density,

$$\begin{aligned} \mathcal{E} &= \partial_0\sigma\partial^0\bar{\sigma} + |\nabla\sigma|^2 + V(\sigma) && (45)\\ &= \omega^2|\psi|^2 + |\nabla\psi|^2 + V(\psi), && (46) \end{aligned}$$

and hence the energy,

$$E = \int d^3x\, \mathcal{E} \quad (47)$$

$$= Q\omega^2 + \int d^3x \left[|\nabla\psi|^2 + V(\psi)\right]. \quad (48)$$

Note: need to be able to construct system where $|\psi| = 0$ everywhere except in a region where $|\psi| \neq 0$. This region will contain all the charge and is the Q-ball. Notice that we can write the energy as

$$E = \frac{Q^2}{\Sigma} + \int d^3x \left[|\nabla\psi|^2 + V(\psi)\right]. \quad (49)$$

Hence, the solution where $|\psi| = 0$ everywhere (i.e. $\Sigma = 0$) has infinite energy (unless $Q = 0$). Therefore, having $Q \neq 0$ guarantees that $|\psi| \neq 0$ and therefore that some configuration exists with non-zero charge density [3].

The idea of charge can be extended to include currents. Recall that the ansatz allowed us to have charge. Suppose we wanted a winding. The ansatz can be

$$\sigma(t, \mathbf{x}) = \psi(r)e^{i(\omega t + kz)}. \quad (50)$$

Thus, k is the wavenumber *along* the z-direction. The energy now becomes

$$E = \frac{1}{\Sigma}\left(Q^2 + k^2\right) + \int d^3x \left[\left(\frac{d\psi}{dr}\right)^2 + V(\psi)\right]. \quad (51)$$

This is the first step in understanding *superconducting cosmic strings*.

C. Vortons

A vorton is a loop of superconducting string.

A simpler version is a kinky vorton: a loop of superconducting domain wall. Take a domain wall and dump an amount of charge on it. The field theory is now quite complicated:

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi + \partial_\mu\sigma\partial^\mu\bar{\sigma} - V(\phi, |\sigma|) \quad (52)$$

Exact solutions exist.

Rough argument (actual argument is a lot more complicated!) Imagine an energy given by

$$E = \alpha R, \quad (53)$$

where R is the radius of a loop and α is a constant. Then the energy is minimized when $R = 0$ (thats pretty obvious). Now suppose

$$E = R + \frac{1}{R}, \quad (54)$$

in which case the energy is minimized when $R = 1$: this will be a object with non-zero size that minimizes the energy: a soliton!

With vortons you want charge and current, so write the σ-field using an ansatz

$$\sigma(t, r, \theta, \varphi) = \psi(r)e^{i(\omega t - N\theta)}, \quad (55)$$

where N an integer and θ the polar coordinate in the (x, y)-plane. This gives rise to an angular momentum

$$J = NQ. \quad (56)$$

As you know, angular momentum is conserved. Therefore, start with some J, always have that J.

Basically, the energy becomes a complicated function of R, N, Q. So can work out what R minimizes the energy for a particular value of N, Q.

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[1] Noethers theorem states that associated with a continuous invariance is a conserved quantity. Invariance of a system under translation in time implies the conservation of energy. Invariance under translation implies conservation of momentum (if under angular translation – i.e. rotation – then angular momentum. Another interesting example is if a theory is invariant under $U(1)$ transformations then there exists a "charge" and "current".

[2] Interpreting Σ as a particle number can be useful in QFT.

[3] There are a load of caveats to this: the potential must be non-renormalizable