

Electromagnetism

Notes taken by J.Pearson, from a S3 course at the U.Manchester.
Lecture delivered by Dr.D.Bailey.
Sept-Dec 06.

July 28, 2007

Contents

| | | |
|----------|--|----------|
| 1 | Electrostatics | 3 |
| 1.1 | Coulomb's Law | 3 |
| 1.2 | Electric Potential | 3 |
| 1.3 | Electric Dipole | 3 |
| 1.4 | Gauss' Law | 3 |
| 1.4.1 | Integral Form | 3 |
| 1.4.2 | Differential Form | 4 |
| 1.5 | Capacitors & Dielectrics | 4 |
| 1.6 | Polarisation | 4 |
| 1.6.1 | Mechanisms for Polarisation: | 4 |
| 1.6.2 | Dipole Moment | 4 |
| 1.7 | Electric Displacement Vector | 5 |
| 1.7.1 | Boundary Conditions | 5 |
| 1.7.2 | Energy Density | 6 |
| 2 | Magnetostatics | 6 |
| 2.1 | Lorentz Force Law | 6 |
| 2.2 | Maxwell's 2nd Equation | 6 |
| 2.3 | Current Density | 6 |
| 2.4 | The Hall Effect | 7 |

| | | |
|----------|--|-----------|
| 2.5 | Force on a Wire | 7 |
| 2.6 | Ampere's Law | 8 |
| 2.6.1 | Integral Form | 8 |
| 2.6.2 | Differential Form | 8 |
| 2.7 | Magnetic Vector Potential | 8 |
| 2.8 | Magnetostatics in Materials | 8 |
| 2.8.1 | Magnetic Dipoles | 8 |
| 2.8.2 | Forms of Magnetisation | 9 |
| 2.9 | Magnetic Field Vector | 9 |
| 3 | Maxwell's Equations | 9 |
| 4 | EM Waves in Vacuum | 10 |
| 4.1 | Poynting Vector | 10 |
| 4.1.1 | Impedance of Free Space | 11 |
| 4.1.2 | Irradiance | 11 |
| 4.1.3 | Radiation Pressure | 11 |
| 5 | Polarisation States | 11 |
| 5.1 | Plane/Linear Polarisation | 11 |
| 5.2 | Circular/Elliptical Polatisation | 11 |
| 6 | EM Waves in Materials | 11 |
| 7 | EM Waves in a Conducting Medium | 12 |
| 7.1 | Plasmas | 13 |

1 Electrostatics

1.1 Coulomb's Law

$$\underline{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \hat{r} \quad (1)$$

1.2 Electric Potential

From $W = q \int_a^b \mathbf{E} \cdot d\mathbf{l}$, we write the potential difference:

$$\phi(r_a) - \phi(r_b) = - \int_a^b \mathbf{E} \cdot d\mathbf{l} \quad (2)$$

And the potential in going from infinity to r :

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r} \quad (3)$$

We have:

$$\mathbf{E} = -\nabla\phi \quad (4)$$

In electrostatic fields only, as \mathbf{E} is conservative, we have $\nabla \times \mathbf{E} = 0$.

1.3 Electric Dipole

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}} \right) \quad (5)$$

For the off-axis potential, for two charges $\pm q$, separated by d .

1.4 Gauss' Law

1.4.1 Integral Form

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} Q_{enc} = \frac{1}{\epsilon_0} \int_V \rho dV \quad (6)$$

Where the closed surface S encloses the volume V .

1.4.2 Differential Form

From the integral form, we use the divergence theorem:

$$\int_S \mathbf{a} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{a} dV \quad (7)$$

$$\Rightarrow \int_S \mathbf{E} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{E} dV = \frac{1}{\epsilon_0} \int_V \rho dV \quad (8)$$

And as we can shrink the volume down to a point, we result in:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (9)$$

This is Maxwell's first equation.

As $\mathbf{E} = -\nabla\phi$, we can also write Poisson's equation, and Laplace's equation (i.e. if no charges present):

$$\nabla^2\phi = \frac{\rho}{\epsilon_0} \quad (10)$$

$$\nabla^2\phi = 0 \quad (11)$$

1.5 Capacitors & Dielectrics

$$C = \frac{Q}{V} \quad (12)$$

Find the potential difference V from $\int_a^b \mathbf{E} \cdot d\mathbf{l}$.

A dielectric material is an insulator: thus conductivity is zero.

Dielectrics gain dipole moments when placed in an electric field. i.e. they polarise. $C = \epsilon C_0$. As dielectric inserted, capacitance increases, thus reducing the internal electric field.

1.6 Polarisation

1.6.1 Mechanisms for Polarisation:

Electronic: The atoms within the dielectric shift, their electron clouds moving, hence leaving the atom with a dipole, which then aligns with the applied electric field.

Aligned Dipoles: Works in a similar way, but this time the intrinsic dipoles of the molecules align - when there is no external field present, they are at 'random'.

1.6.2 Dipole Moment

The dipole moment \mathbf{p} due to two charges q separated by \mathbf{d} is given by:

$$\mathbf{p} = q\mathbf{d} \quad (13)$$

The total polarisation \mathbf{P} is therefore given by:

$$\mathbf{P} = n\mathbf{p} \quad (14)$$

basically the total number of dipole moments.

The surface charge density is given by the dot product of the polarisation with the outward normal of the surface:

$$\sigma_p = \mathbf{P} \cdot \hat{\mathbf{n}} \quad (15)$$

We have the following relations between polarisation \mathbf{P} , electric field \mathbf{E} , electric susceptibility χ_E & polarisation charge density ρ_p :

$$\mathbf{P} = (\epsilon - 1)\epsilon_0\mathbf{E} \quad (16)$$

$$\chi_E = \epsilon - 1 \quad (17)$$

$$\rho_p = -\nabla \cdot \mathbf{P} \quad (18)$$

In writing (16), we assumed: linearity, isotropy, homogeneity & non-conducting.

1.7 Electric Displacement Vector

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \quad (19)$$

Thus, we have a new version of Gauss' law:

$$\nabla \cdot \mathbf{D} = \rho_f \quad (20)$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_f dV \quad (21)$$

Where ρ_f denotes free charges.

\mathbf{D} is not affected by polarisation charges, and \mathbf{P} exists only within the dielectric.

The total charge density is the sum of the free charge density with the polarisation charge density:

$$\rho = \rho_p + \rho_f \quad (22)$$

1.7.1 Boundary Conditions

D_{perp} is continuous across boundaries.

E_{\parallel} is continuous across boundaries.

That is:

$$D_1 \cos \theta_1 = D_2 \cos \theta_2 \quad (23)$$

$$E_1 \sin \theta_1 = E_2 \sin \theta_2 \quad (24)$$

1.7.2 Energy Density

Can be given by:

$$U = \int_V \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \, d\tau \quad (25)$$

2 Magnetostatics

2.1 Lorentz Force Law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (26)$$

So, if $\mathbf{E} = 0 \Rightarrow \mathbf{F} = q\mathbf{v} \times \mathbf{B}$.

So, if a charge is moving solely in the presence of a magnetic field, we have \mathbf{F} being perpendicular to velocity, hence motion in a circle, with period τ :

$$F = \frac{mv^2}{r} = qvB \quad (27)$$

$$\tau = \frac{2\pi r}{v} = \frac{2\pi m}{qB} \quad (28)$$

Hence, we define the cyclotron frequency $\nu = \frac{1}{\tau}$ as:

$$\nu = \frac{qB}{2\pi m} \quad (29)$$

2.2 Maxwell's 2nd Equation

$$\nabla \cdot \mathbf{B} = 0 \quad (30)$$

2.3 Current Density

Current density is the number of charges, times the charge, times the drift velocity of the charges. Alternatively, it is the current per unit area:

$$j = \frac{I}{A} = -Nev_d \quad (31)$$

$$\Rightarrow I = \int_S \mathbf{j} \cdot d\mathbf{A} \quad (32)$$

Ohm's law becomes:

$$\mathbf{j} = \sigma \mathbf{E} \quad (33)$$

Where σ is electrical conductivity.

2.4 The Hall Effect

Current \mathbf{I} flows along a slab of conductor, with width w and depth d - hence area A . A magnetic field \mathbf{B} intersects the slab, so that \mathbf{B} is perpendicular to \mathbf{I} . Hence, the charge carriers 'feel' a force. The charge carriers are electrons, with drift velocity v_d .

We define:

$F_m \equiv$ magnetic force on negative charge carriers;

$F_e \equiv$ electric force on charge build up.

So, we begin by writing the force, and current:

$$F_e = eE \quad (34)$$

$$F_m = ev_d B \quad (35)$$

$$I = nev_d A \quad (36)$$

$$\Rightarrow v_d = \frac{I}{neA} \quad (37)$$

$$\Rightarrow F_m = \frac{eIB}{neA} \quad (38)$$

And, to find the potential in equilibrium (i.e. $F_e = F_m$):

$$F_e = eE = \frac{V_H e}{w} \quad (39)$$

Where the potential is $V_H = wE$. Thus, the system is in equilibrium when:

$$\frac{V_H e}{w} = \frac{eIB}{neA} \quad (40)$$

Therefore, the potential set up, the 'Hall Voltage' in equilibrium, across the slab is:

$$V_H = \frac{IB}{ned} \quad (41)$$

2.5 Force on a Wire

Force on one electron, by Lorentz:

$$\mathbf{f} = -e\mathbf{v} \times \mathbf{B} \quad (42)$$

If the conductor has length $d\mathbf{l}$, and area \mathbf{A} ; thus, $nAdl$ electrons, at a velocity v_d . Hence, the total force:

$$d\mathbf{F} = -(nAdl)e\mathbf{v}_d \times \mathbf{B} \quad (43)$$

But, $I = -neAv_d$. Thus:

$$d\mathbf{F} = Id\mathbf{l} \times \mathbf{B} \quad (44)$$

2.6 Ampere's Law

2.6.1 Integral Form

$$\oint_{\ell} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \sum I \quad (45)$$

2.6.2 Differential Form

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad (46)$$

If the integral version is used on a long straight wire, of radius r :

$$\oint_{\ell} \mathbf{B} \cdot d\mathbf{l} = 2\pi r B = \mu_0 I \quad (47)$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r} \quad (48)$$

2.7 Magnetic Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (49)$$

2.8 Magnetostatics in Materials

Inductance in a vacuum is $L_0 = \mu_0 N^2 \pi r^2 l$, and in materials is $L = \mu L_0$; where $\mu = \frac{1}{1-\chi_B}$.

2.8.1 Magnetic Dipoles

Magnetic moment = current \times area:

$$\mathbf{m} = I\mathbf{A} \quad (50)$$

Torque on a current loop:

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B} \quad (51)$$

Potential energy of current loop:

$$U(\theta) = -\mathbf{m} \cdot \mathbf{B} \quad (52)$$

Magnetisation is like polarisation: total dipole moment per unit area:

$$\mathbf{M} = N\mathbf{m} \quad (53)$$

Thus, we have:

$$\mathbf{M} = \chi_B \frac{\mathbf{B}}{\mu_0} \quad (54)$$

2.8.2 Forms of Magnetisation

Diamagnetism: From individual atoms, and electrons orbiting in \mathbf{B} fields. Weak. Linear. Negative.

Paramagnetism: From moments aligning. Stronger. Linear. Positive.

Ferromagnetism: Due to domains within material. Strong. Non-linear. Permanent. Has hysteresis.

Magnetic dipoles in materials can be visualised as small current loops, where the internal components cancel out, to leave only a surface current.

As before, we have:

$$\mathbf{i}_s = \mathbf{M} \times \hat{\mathbf{n}} \quad (55)$$

$$\mathbf{i}_b = \nabla \times \mathbf{M} \quad (56)$$

$$\Rightarrow \mathbf{i} = \mathbf{i}_s + \mathbf{i}_b \quad (57)$$

$$\mathbf{B} = \mathbf{B}_0 + \mu_0 \mathbf{M} \quad (58)$$

2.9 Magnetic Field Vector

We have:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad (59)$$

Thus, we have a new Ampere's law, involving free-currents:

$$\nabla \times \mathbf{H} = \mathbf{j}_f \quad (60)$$

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_f \quad (61)$$

Energy in magnetic fields is thus:

$$U = \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \quad (62)$$

\mathbf{H} -field lines can be discontinuous:

$$\nabla \cdot \mathbf{H} = -\nabla \times \mathbf{M} \quad (63)$$

And, at boundaries we have that B_{perp} and H_{\parallel} are conserved.

3 Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (64)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (65)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (66)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (67)$$

With corresponding alternatives:

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \rho_f & (68) \\ \nabla \times \mathbf{H} &= \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t}\end{aligned}$$

4 EM Waves in Vacuum

Maxwell's equations in free space read:

$$\nabla \cdot \mathbf{E} = 0 \quad (70)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (71)$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (72)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (73)$$

We can easily derive the wave equation:

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (74)$$

Where the speed is given by $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$. (74)

Now, solutions to wave equations have the form $f(z - vt)$; thus we can write things like plane wave solutions $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$.

We write the relation between \mathbf{E} and \mathbf{B} :

$$\mathbf{B} = \frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E} \quad (76)$$

$$\Rightarrow cB_y = E_x \quad (77)$$

Energy density in the wave is given by:

$$U = \frac{1}{2} \int_V (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) d\tau \quad (78)$$

Or, equivalently:

$$U = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \quad (79)$$

$$= \epsilon_0 E_0^2 \cos^2(kz - \omega t) \quad (80)$$

Hence, electric and magnetic fields carry the same amount of energy.

4.1 Poynting Vector

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \mathbf{E} \times \mathbf{H} \quad (81)$$

$$\Rightarrow S = \frac{1}{\mu_0 c} E_0^2 \quad (82)$$

4.1.1 Impedance of Free Space

$$= \left(\sqrt{\frac{\epsilon_0}{\mu_0}} \right)^{-1} = 377\Omega \quad (83)$$

4.1.2 Irradiance

A time averages Poynting vector:

$$I = \langle \mathbf{S} \rangle = \frac{E_0^2}{2\mu_0 c} \quad (84)$$

4.1.3 Radiation Pressure

If radiation absorbed or reflected:

$$P^{abs} = \frac{I}{c} \quad (85)$$

$$P^{ref} = \frac{2I}{c} \quad (86)$$

So, the force exerted due to radiation pressure is $F = PA$, the standard pressure times area.

5 Polarisation States

5.1 Plane/Linear Polarisation

Electric field is confined to a plane:

$$\mathbf{E} = E_0 \cos(kz - \omega t) \mathbf{i} \quad (87)$$

5.2 Circular/Elliptical Polatisation

$$\mathbf{E} = E_x \cos(kz - \omega t) \mathbf{i} + E_y \cos(kz - \omega t + \delta) \mathbf{j} \quad (88)$$

If $E_x = E_y$ & $\delta = \pm \frac{\pi}{2}$, then the wave is circularly polarised. Anything else gives elliptical polarisation.

6 EM Waves in Materials

Here we have $\epsilon_0 \rightarrow \epsilon_0 \epsilon$ and $\mu_0 \rightarrow \mu \mu_0$. We still assume no charges/currents. The wave equation becomes:

$$\nabla^2 \mathbf{E} = \mu \mu_0 \epsilon \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (89)$$

Now, the velocity of the wave is:

$$v = \frac{1}{\sqrt{\mu\mu_0\epsilon\epsilon_0}} = \frac{c}{\sqrt{\mu\epsilon}} = \frac{c}{n} \quad (90)$$

$$\Rightarrow n = \sqrt{\mu\epsilon} \approx \sqrt{\epsilon} \quad (91)$$

We have Snell's law: $n_1 \sin \theta_1 = n_2 \sin \theta_2$, but generally is:

$$k_I \sin \theta_I = k_R \sin \theta_R = k_T \sin \theta_T \quad (92)$$

For an incident wave \mathbf{E}_I , with reflection \mathbf{E}_R , and transmission \mathbf{E}_T . (92)

Now, boundary conditions state:

$$E_I + E_R = E_T \quad (94)$$

$$H_I + H_R = H_T \quad (95)$$

$$\Rightarrow \frac{B_I}{\mu_0} - \frac{B_R}{\mu_0} = \frac{B_T}{\mu\mu_0} \quad (96)$$

But, we also have that $B = \frac{1}{c}E = \sqrt{\epsilon_0\mu_0}E$, thus:

$$E_I \sqrt{\frac{\epsilon_0}{\mu_0}} - E_R \sqrt{\frac{\epsilon_0}{\mu_0}} = E_T \sqrt{\frac{\epsilon\epsilon_0}{\mu\mu_0}} \quad (97)$$

Which, using the approximation that $\mu \approx 1$:

$$E_I - E_R = nE_T \quad (98)$$

Thus, we define the reflection and transmission coefficients thus:

$$R = \frac{E_R^2}{E_I^2} = \left(\frac{1-n}{n+1}\right)^2 \quad (99)$$

$$T = \frac{E_T^2 v}{E_I^2 c} = \frac{4n}{(1+n)^2} \quad (100)$$

Noticing that $T + R = 1$

7 EM Waves in a Conducting Medium

Here we have $\mathbf{j} = \sigma\mathbf{E}$. Thus, with this expression for current density, Maxwell 3 becomes:

$$\nabla \times \mathbf{B} = \mu_0\sigma\mathbf{E} + \epsilon_0\mu_0 \frac{\partial\mathbf{E}}{\partial t} \quad (101)$$

However, in a 'good conductor', we have that $\mu_0\sigma\mathbf{E} \gg \epsilon_0\mu_0 \frac{\partial\mathbf{E}}{\partial t}$. Hence, we have:

$$\nabla \times \mathbf{B} = \mu_0\sigma\mathbf{E} \quad (102)$$

We can also derive the skin depth attenuation factor: $\delta = \sqrt{\frac{2}{\mu_0\sigma\omega}}$.

7.1 Plasmas

Using: $\mathbf{F} = m\ddot{\mathbf{r}} = q\mathbf{E}$ and $\mathbf{j} = Ne\dot{\mathbf{r}}$, we can derive:

$$\nabla^2\mathbf{E} = \mu_0 n_e e \left(\frac{e\mathbf{E}}{m_e} \right) + \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (103)$$

From which we can derive that below the plasma frequency $\omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}$, incident EM waves are strongly attenuated.