Function Decompositions
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Here we shall look at the decomposition of functions, into a basis. We shall start with a very simple case: money.

Let us suppose that we are working with money; in denominations:

$$1p, 2p, 5p, 10p, 20p \dots$$

This set of coins forms something that we will end up calling our *basis*.
So, in everyday life, suppose that one had, in their wallet, a collection of coins, up to 20pence pieces (i.e. we shall not consider higher denomination than 20p, just for brevity). Now, consider that we have the rule that we must pay for things, using the maximum denomination of coin possible. For example, suppose something cost 22p, then, one would use one each of 20p,2p. Of course, one could make 22p in other ways (such as two 10p, and two 1p), but we shall ignore that possibility.
Ok, so suppose that we have some total amount, T, to pay; and to do so, we must make up some combination of coins. We shall, in general, use various amounts of different denomination of coins. To continue, let us begin to use some notation: The set of coins (i.e. the denominations possible) are called a *basis*, and are denoted by $\{C_i\}$. In this notation, we have:

$$C_1 = 1p \quad C_2 = 2p \quad C_3 = 5p \quad C_4 = 10p \quad C_5 = 20p$$

So, in a more compact notation, we have that the set of coins can be written like:

$$\{C_i\}_1^n$$

This notation should be read in the following way: “the set of coins, C_i, where i runs from 1 to n”. Thus, in our simple basis (i.e. our $1 \rightarrow 20p$ coin set), $n = 5$. This upper limit on denomination is actually a restriction, of sorts. We will discuss it later. However, for completeness, supposing that $n = 6$, then the extra ‘element’ of the set $\{C_i\}$ is just $C_6 = 50p$.

Now, consider some total amount $T = 62p$. Then, we can make that total amount (under the $n = 5$ ‘restriction’) in the following way:

$$T = 62p = (3 \times 20p) + (1 \times 2p) = 3C_5 + 1C_2$$

So, we see that we can make some amount T that is not one of the denominations, by creating some sort of sum over the other coins. Now then, for generality, suppose that some arbitrary amount T can be made from a sum of an arbitrary number of coins of any denomination (including zero number of a particular denomination). Thus, mathematically, this is:

$$T = a_1C_1 + a_2C_2 + a_3C_3 + a_4C_4 + a_5C_5$$

Or, in a more compact notation:

$$T = \sum_{i=1}^{n} a_iC_i$$

The number (or coefficient) a_i is the number of that particular coin used in the sum; in another 'language' this is the amplitude of the basis element used. For example, considering our total $T = 62p$, from above, we have:

$$a_1 = 0 \quad a_2 = 1 \quad a_3 = 0 \quad a_4 = 0 \quad a_5 = 3$$

We see that the coefficient a_i is the number of times that the basis C_i is used to make the total "work", or come out of the other end.

Now, one of the things that we started out by saying, is that we must use the highest denomination possible, in a sum. That is, instead of using 2 1p's, we use 1 2p. This arises because the basis (i.e. the coins) themselves can be made as a sum over others. For example, to make 20p, one can use 4 5p's:

$$C_5 = 20p = 4 \times C_3$$

So, some basis C_j may be made from a sum over other basis:

$$C_j = \sum_i a_i C_i$$

This gets very messy, very quickly; and dosent really help our intuition develop in terms of function decomposition. However, that a basis cannot be made out of a sum over other bases, gives the basis a special property: orthogonality. Orthogonality is the statement that a basis C_i can not be made from any other basis. So, in the above sum, the only term that contributes is when $i = j$, and we must have that $a_j = 1$. A mathematical way of writing this, is using the Kronecker-delta:

$$C_j = \sum_i a_i C_i \frac{\delta_{ij}}{a_j}$$

Where we have also divided by some coefficient, with the label j. So, the Kronecker-delta has the property that it is zero everywhere that $i \neq j$. And when $i = j$ it has the value of unity. So, the sum on the RHS will sweep over all values of i, but will only return a non-zero value for $i = j$:

$$C_j = a_j C_j \frac{\delta_{jj}}{a_j} = C_j$$

So then, let us bring together our notation, and words for things.
We write $\{C_i\}$ to denote a basis set: the things that the total is summed over. However, to get to the total, we must use different numbers of each basis; the number of each basis used is some coefficient a_i. We also have that the basis has some sort of upper limit: there is only a finite number of denomination of coins.

Now then, let us consider the decomposition of a function, into some basis. Let us say that we have some arbitrary function $f(x)$, and that we are decomposing it into a basis v_i, with some coefficient:

$$f(x) = \sum_{i=1}^{n} a_i v_i$$

So, in the language of our coins previously, we are making some total amount out of a set of predetermined values, using different amounts of each value (or basis). We shall now start to use the word "basis set" to denote the set of basis vectors $\{v_i\}$. A single 'basis vector' is thus just v_i. Now, in mathematics, there are a fair few basis vectors to choose from. Some require that the sum be taken up to infinity to decompose a function fully, some may only require a finite amount.

It is not the intention of this document to go into the types of function that one may use as the basis. Regardless, one can decompose functions using the Fourier basis of sines & cosines; or one may use spherical harmonics to decompose a function. The basis one chooses is usually chosen because of the system of interest. If the system is on the surface of a sphere, or has spherical symmetry, then one tends to use spherical harmonics:

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta,\phi)$$

Here, we have that there is some coefficient $a_{\ell m}$ which determines the "amount" of the spherical harmonic basis to use.