

## Hamilton's Equations:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

Putting:

$$\frac{\partial L}{\partial \dot{q}_i} = p_i$$

$$\therefore \frac{dp_i}{dt} - \frac{\partial L}{\partial q_i} = 0$$

$$\therefore \dot{p}_i = \frac{\partial L}{\partial q_i}$$

Now:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} L(q, \dot{q}, t) = f_i(q, \dot{q}, t)$$

There are  $n$   $p_i$ 's and  $n$   $\dot{q}_i$ 's. $\therefore$  solving for  $\dot{q}_i$  in terms of the other variables:

$$\therefore \dot{q}_i = g_i(q, p, t) \quad (1)$$

Let:

$$H = \sum_{i=1}^n p_i \dot{q}_i - L \quad (2)$$

From (1) and (2):

$$H = H(p, q, t)$$

When expressed in this form,  $H$  is called the Hamiltonian.

Using the summation convention:

$$H = H(p, q, t) = p_i \dot{q}_i - L \quad (3)$$

## Differentials:

Taylor series:

$$\begin{aligned} f(a_1 + h_1, a_2 + h_2, \dots, a_n + h_n) &= f(a_1, a_2, \dots, a_n) + h_1 \frac{\partial f}{\partial a_1} + h_2 \frac{\partial f}{\partial a_2} + \dots + h_n \frac{\partial f}{\partial a_n} \\ &+ \frac{1}{2!} \left( \frac{\partial^2 f}{\partial a_1^2} h_1^2 + \frac{\partial^2 f}{\partial a_2^2} h_2^2 + \dots + \frac{\partial^2 f}{\partial a_n^2} h_n^2 + \right. \\ &\left. \frac{\partial^2 f}{\partial a_1 \partial a_2} h_1 h_2 + \frac{\partial^2 f}{\partial a_1 \partial a_3} h_1 h_3 + \dots \right) \end{aligned}$$

That is:

$$\begin{aligned} f(a_1 + h_1, a_2 + h_2, \dots, a_n + h_n) &= f(a_1, a_2, \dots, a_n) + \sum_{i=1}^n \frac{\partial f}{\partial a_i} h_i + \frac{1}{2!} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial a_i \partial a_j} h_i h_j \end{aligned}$$

$$+ \frac{1}{3!} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \frac{\partial^3 f}{\partial a_i \partial a_j \partial a_k} h_i h_j h_k$$

+ ...

$$\therefore f(a_1 + h_1, a_2 + h_2, \dots, a_n + h_n) - f(a_1, a_2, \dots, a_n) = \sum_{i=1}^n \frac{\partial f}{\partial a_i} h_i + \dots$$

Let:

$$f(a_1 + h_1, a_2 + h_2, \dots, a_n + h_n) - f(a_1, a_2, \dots, a_n) = \mathbf{df}$$

$$h_i = \mathbf{da}_i \quad i = 1, 2, \dots, n$$

Then:

$$\mathbf{df} = \sum_{i=1}^n \frac{\partial f}{\partial a_i} \mathbf{da}_i + \dots$$

So, to the 1<sup>st</sup> order of “small quantities”:

$$\mathbf{df} \approx \sum_{i=1}^n \mathbf{da}_i \frac{\partial f}{\partial a_i}$$

We write:

$$df = \sum_{i=1}^n da_i \frac{\partial f}{\partial a_i}$$

$df, da_i$  are called differentials.

$f(a_1, a_2, \dots, a_n)$  could also be written as  $f(x_1, x_2, \dots, x_n)$ , so:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i = \frac{\partial f}{\partial x_i} dx_i$$

Now, going back to equation (3), and using the differential formulation:

$$\begin{aligned} dH &= d(p_i, \dot{q}_i) - dL \\ &= \dot{q}_i dp_i + p_i d\dot{q}_i - \left( \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt \right) \\ &= \dot{q}_i dp_i + p_i d\dot{q}_i - \left( \frac{\partial L}{\partial q_i} dq_i + p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt \right) \\ &= \dot{q}_i dp_i - \left( \dot{p}_i dq_i + \frac{\partial L}{\partial t} dt \right) \quad (4) \end{aligned}$$

As:

$$\begin{aligned} &H(p, q, t) \\ \therefore dH &= \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt \\ \therefore \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt &= \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt \end{aligned}$$

Now, the coefficients are equal, so:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad (5)$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \quad (6)$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad (7)$$

From (4):

$$\begin{aligned} \frac{dH}{dt} &= \dot{q}_i \dot{p}_i - \dot{p}_i \dot{q}_i - \frac{\partial L}{\partial t} \\ \therefore \frac{dH}{dt} &= -\frac{\partial L}{\partial t} \end{aligned} \quad (8)$$

Suppose that  $L$  does not contain the time explicitly, i.e.:  $L(q, \dot{q})$ . Then:

$$\frac{\partial L}{\partial t} = 0$$

$\therefore$  from (8):

$$\frac{dH}{dt} = 0 \quad (9)$$

$\therefore H$  is a constant.

But:

$$\begin{aligned} H &= p_i \dot{q}_i - L \\ &= \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \\ &= \dot{q}_i \frac{\partial}{\partial \dot{q}_i} (T - V) - (T - V) \\ &= \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - (T - V) \\ &= 2T - (T - V) \\ &= T + V \end{aligned}$$

$\therefore$

$$T + V = \text{const}$$

### Small Oscillations:

From before:

$$T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j$$

We are going to take  $\dot{q}_i$  to be small.

$$V = V(q) = V(0,0,\dots,0) + \left. \frac{\partial V}{\partial q_i} q_i \right|_{q_i=0} + \frac{1}{2!} \left. \frac{\partial^2 V}{\partial q_i \partial q_j} q_i q_j \right|_{q_i=0} + \dots$$

If the body is in equilibrium at  $q_i = 0$ , then:

$$\left. \frac{\partial V}{\partial q_i} \right|_0 = 0$$

Putting:

$$V(0,0,\dots,0) = c$$

So that:

$$V - c = \frac{1}{2!} \frac{\partial^2 V}{\partial q_i \partial q_j} q_i q_j \Big|_{q=0} + \dots$$

Replacing  $V - c$  by  $V$ , then:

$$V = \frac{1}{2} b_{ij} q_i q_j$$

Where:

$$b_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j} \Big|_0 = \text{const}$$

$$a_{ij} = a_{ij}(q) = a_{ij}(0,0,\dots,0) + \frac{\partial a_{ij}}{\partial q_k} q_k \Big|_0 + \dots$$

$\therefore$  to the second order in  $q_i$ 's we have:

$$T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j$$

$\therefore$  we can write, to the second order of small quantities:

$$T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j$$

$$V = \frac{1}{2} b_{ij} q_i q_j$$

Where  $a_{ij}, b_{ij}$  are constants.

Lagrange equations are:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = - \frac{\partial V}{\partial q_i}$$

$$T = \frac{1}{2} a_{kl} \dot{q}_k \dot{q}_l$$

$$\therefore \frac{\partial T}{\partial \dot{q}_i} = \frac{1}{2} a_{kl} \left( \frac{\partial \dot{q}_k}{\partial \dot{q}_i} \dot{q}_l + \frac{\partial \dot{q}_l}{\partial \dot{q}_i} \dot{q}_k \right)$$

$$= \frac{1}{2} (\mathbf{d}_{ki} \dot{q}_l + \mathbf{d}_{li} \dot{q}_k)$$

$$= \frac{1}{2} (a_{il} \dot{q}_l + a_{ik} \dot{q}_k)$$

$$= \frac{1}{2} (2a_{ik} \dot{q}_k)$$

$$= a_{ik} \dot{q}_k$$

$$\frac{\partial T}{\partial q_i} = 0$$

$$\frac{\partial V}{\partial q_i} = \frac{1}{2} (2b_{ij} a_j) = b_{ij} a_j$$

$\therefore$  Lagrange gives:

$$a_{ik} \ddot{q}_k = -b_{ij} q_j$$

Or:

$$a_{ik}\ddot{q}_k + b_{ik}q_k = 0$$

Let:

$$q_k = A_k \cos(\omega t + \epsilon)$$

$\therefore$

$$\ddot{q}_k = -\omega^2 A_k \cos(\omega t + \epsilon)$$

So:

$$(-a_{ik}\omega^2 A_k + b_{ik}A_k)\cos(\omega t + \epsilon) = 0$$

$\therefore$

$$-a_{ik}\omega^2 A_k + b_{ik}A_k = 0$$

$$(a_{ij}\omega^2 - b_{ij})X = 0 \quad X = \begin{pmatrix} A_1 \\ A_2 \\ \dots \\ A_n \end{pmatrix}$$

$\therefore X$  is an eigenvector and  $\omega^2$  is an eigenvalue of the matrix:

$$a_{ij}\omega^2 - b_{ij}$$

$\therefore$

$$|a_{ij}\omega^2 - b_{ij}| = 0$$