

Matrices:

Suppose you have two simultaneous equations:

$$\begin{aligned} 2x + y &= 11 \\ 4x + 5y &= 31 \end{aligned} \tag{1}$$

Now, if you see that x and y are present in both equations, you may write:

$$\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 31 \end{pmatrix} \tag{2}$$

You should be able to see where the numbers have come from.

If you think back to vector notation, you should remember that you can express $\begin{pmatrix} x \\ y \end{pmatrix}$ as a vector \underline{x} say.

Now, you can express a *matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as $\underline{\underline{A}}$.

So, you can express (2) as:

$$\underline{\underline{A}}\underline{x} = \underline{x'}$$

Infact, a vector like x is a matrix, just with fewer columns.

More generally however, you would say that the matrix has i rows, and j columns. And you can find elements within a matrix like this – if you want the element that is on the 2nd row, 1st column, you would write A_{ij} , and in the case above, $A_{21} = b$. Its quite important to get the order right – it goes along the top, then down – 2 across, 1 down.

And in the matrix in equation (2), $A_{22} = 5$ (2 across, 2 down!)

The general 2x2 matrix (2 columns, 2 rows) can then be written as $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

So, if you use some general functions, or numbers if you like, in equations (1):

$$\begin{aligned} ax + by &= x' \\ cx + dy &= y' \end{aligned}$$

You should be able to express this in the form of (2):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \text{or} \quad \underline{\underline{A}}\underline{x} = \underline{x'}$$

This can be thought of as a set of operations (the matrix) acting upon some points (the vector) to give some new points (the RHS vector).

This is a transformation matrix.

One set of coordinates going to another set.

Now, if you were to plot the vector $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ on the x-y plane [so, you get a point at coordinates (x, y)]; then the vector $\underline{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$, you will begin to see a physical representation of what the matrix is doing – or transforming.

You may see matrices written as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, rather than curly brackets. It doesn't matter.

If you think back to the original equations, and how we got a matrix from simultaneous equations, you can start making up some algebra for these things. In particular, multiplication.

If you go from:

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 15 \\ 42 \end{pmatrix}$$

You can see that the top row of the matrix \underline{A} is “going down” the vector. 1 is multiplying 3. And 2 multiplying 6. then you add up these products:

$$(1 \times 3) + (2 \times 6) = 15$$

Similarly for the second row:

$$(4 \times 3) + (5 \times 6) = 42$$

In this example, the matrix is transforming the coordinates $(3, 6)$ to the point $(15, 42)$. If you draw it on an x-y plane it will become a lot clearer!

If you think a bit more about vectors, you will remember that $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a 3D vector.

It turns out there are also 3x3 matrices!

So, using the same notation, with a general matrix, and a general vector:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \text{or:} \quad \underline{Ax} = \underline{x}'$$

Again, if you think about some simultaneous equations:

$$ax + by + cz = x'$$

$$dx + ey + fz = y'$$

$$gx + hy + iz = z'$$

So you can see how to go from equations to matrices.

Note also, that the 3x3 matrix is still just transforming points (x, y, z) to (x', y', z') .
 The notation of A_{ij} before becomes a lot more useful... for example...

$$A_{12} = b \text{ and } A_{31} = g \text{ and } A_{32} = h, \text{ finally } A_{33} = i$$

Across the top, then down.

Here is a numbers example again:

$$\begin{pmatrix} 1 & 2 & 5 \\ 4 & 3 & 2 \\ 8 & 9 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \\ 27 \end{pmatrix}$$

If I write out everything again:

$$\begin{aligned} (1 \times 1) + (2 \times 1) + (5 \times 1) &= 8 \\ (4 \times 1) + (3 \times 1) + (2 \times 1) &= 9 \\ (8 \times 1) + (9 \times 1) + (10 \times 1) &= 27 \end{aligned}$$

You see where the numbers come from.

You can multiply matrices together, and add them... and all sorts! Things I will go into later.

There are a number of useful operations to do to a matrix:

The Determinant, Transpose and the Inverse.

The Transpose:

Suppose we have a matrix $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and then the transpose = $\underline{\underline{A}}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

The A_{ij} notation for this operation is: $A_{ij}^T = A_{ji}$.

Basically, the rows become columns, and the columns rows.

See this for a 3x3 matrix:

$$\underline{\underline{A}} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{Then} \quad \underline{\underline{A}}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

The superscript of T means the transpose of the matrix A .

And example with numbers:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 & 5 \\ 4 & 3 & 2 \\ 8 & 9 & 10 \end{pmatrix} \Rightarrow \underline{\underline{A}}^T = \begin{pmatrix} 1 & 4 & 8 \\ 2 & 3 & 9 \\ 5 & 2 & 10 \end{pmatrix}$$

This may seem a little abstract, and nonsensical, but there will be need to do this a little later.

The Determinant:

Start, again, with the matrix $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Now, the determinant is $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

To calculate this:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a \times d) - (b \times c)$$

It's a little more complex with 3x3 matrices... although you have met them in evaluating cross products:

$$\underline{\underline{A}} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Note the minus sign!!

So, to find the determinant of a 3x3 matrix, you will need to work out the determinant of 3 2x2 matrices.

And, for 4x4 matrices, its basically the same – det's of 4 3x3's, each of which need 3 2x2's! So it gets very messy, very quickly!

A way of thinking about how to write them down:

1st element, then multiply by the determinant of the matrix which DOES NOT INCLUDE the row or column of the element.

But you just need to remember the minus sign for the middle determinant.

The Inverse.

This is the most useful, so far.

Suppose we have simultaneous equations:

$$\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 31 \end{pmatrix}$$

How can we find what x and y are...?

Now, remember that we wrote:

$$\underline{\underline{A}}x = x'$$

Matrices

It is valid to say that:

$$\underline{x} = \underline{A}^{-1} \underline{x}'$$

Where \underline{A}^{-1} is the inverse of \underline{A} .

It turns out that:

$$\underline{A}^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Swap the elements on the leading diagonal, and multiply the others by -1

A numerical example is best:

$$\underline{A} = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} \quad \text{and} \quad |A| = \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} = (2 \times 5) - (1 \times 4) = 6$$

So:

$$\underline{A}^{-1} = \frac{1}{6} \begin{pmatrix} 5 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{6} \\ -\frac{4}{6} & \frac{2}{6} \end{pmatrix}$$

Now, to apply this... to solve the simultaneous equations

$$\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 31 \end{pmatrix}$$

$$\underline{Ax} = \underline{x}' \quad \text{so} \quad \underline{x} = \underline{A}^{-1} \underline{x}'$$

Therefore:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 11 \\ 31 \end{pmatrix}$$

We just calculated the inverse:

$$\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{6} \\ -\frac{4}{6} & \frac{2}{6} \end{pmatrix}$$

So:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{6} \\ -\frac{4}{6} & \frac{2}{6} \end{pmatrix} \begin{pmatrix} 11 \\ 31 \end{pmatrix}$$

And we know how to multiply matrices by vectors:

$$\begin{pmatrix} \frac{5}{6} & -\frac{1}{6} \\ -\frac{4}{6} & \frac{2}{6} \end{pmatrix} \begin{pmatrix} 11 \\ 31 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

So,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{6} \\ -\frac{4}{6} & \frac{2}{6} \end{pmatrix} \begin{pmatrix} 11 \\ 31 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Which is the same as writing:

$$\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}$$

So, we have just found the solutions to the simultaneous equations. It may seem long winded, but it is more useful with 3 simultaneous equations, in 3 unknowns.

The inverse for a 3x3 matrix is a little different: Infact, this is an example of the more general case.

It involves the concept of cofactors:

This also needs you to bear in mind this ‘chess board’ effect:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The cofactor of the top left element - A_{11} - is given by:

$$C_{11} = \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix}$$

There is a cofactor for each element in the 3x3 matrix – so 9 altogether. They are found by the determinant of the matrix containing the elements from the original matrix which are NOT in the row/column containing the cofactor.

So, for example: C_{22} is formed from the elements NOT in the 2nd row OR 2nd column.

The sign is determined from the above chessboard rule.

Now:

The inverse of a 3x3 matrix is:

$$\underline{\underline{A}}^{-1} = \frac{1}{|\underline{\underline{A}}|} \underline{\underline{C}}^T = \frac{1}{|\underline{\underline{A}}|} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

Where $\underline{\underline{C}}^T$ is the transpose of the matrix containing the cofactors of $\underline{\underline{A}}$.

So, in the i/j notation:

$$C_{ij} = \text{cof}(A_{ij}) \text{ and } C_{ij}^T = C_{ji}.$$

Matrix Algebra:

Matrix addition/subtraction is pretty simple:

If you have a matrix equation $\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{C}}$, and you need to find $\underline{\underline{C}}$, then the elements of $\underline{\underline{C}}$ come from adding the corresponding elements of $\underline{\underline{A}}$ and $\underline{\underline{B}}$.

So, $C_{ij} = A_{ij} + B_{ij}$, which will come a lot more transparent if you were to write the elements out.

Similarly for subtraction: $C_{ij} = A_{ij} - B_{ij}$, if $\underline{\underline{A}} - \underline{\underline{B}} = \underline{\underline{C}}$.

Also, when multiplying a matrix by a number (or a scalar!) – which is something we have already used – just multiply every element within the matrix by that number:

$$k\underline{\underline{A}} = kA_{ij}.$$

Multiplication of matrices:

Suppose we want to evaluate:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

The way to do it is:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

What you are doing is:

Divide row one onto column one – multiplying, then adding... that gives the first element.

Divide row one onto column two.

Row two onto column one.

Row two onto column two.

The i/j notation for this is:

$$C_{ij} = \sum_{k=1}^m A_{ik} B_{kj} = A_{ik} B_{kj}$$

Einstein summation convention drops the sum sign.

Note, you can only multiply matrices which have same number of columns, then rows.

The order of multiplication is VERY important.

To multiply 3x3 matrices, write them out, term by term, and evaluate all the products and sums.

An important matrix, which hasn't been mentioned so far, is the identity matrix. It can be expressed as a 2x2, 3x3, 4x4... matrix. It is denoted by $\underline{\underline{I}}$.

$$\underline{\underline{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

You should see the pattern.

Eigenvalues and Eigenvectors.

Consider:

$$\underline{\underline{A}}\underline{x} = \underline{\underline{I}}\underline{x}$$

What that is saying is this:

A matrix is operating on a vector, to give another vector, which is just a scalar multiple of itself.

The vector \underline{x} is then called an eigenvector, and the scalar $\underline{\underline{I}}$ is called the eigenvalue.

Now, this matrix equation, $\underline{\underline{A}}\underline{x} = \underline{\underline{I}}\underline{x}$, can be written as:

$$(\underline{\underline{A}} - \underline{\underline{I}})\underline{x} = 0 \quad \text{using the identity matrix above.}$$

Now, this only has non trivial solutions if:

$$|\underline{\underline{A}} - \underline{\underline{I}}| = 0$$

For an example of 2x2 matrices:

$$\begin{aligned} \underline{\underline{I}}\underline{\underline{I}} &= \underline{\underline{I}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \underline{\underline{I}} & 0 \\ 0 & \underline{\underline{I}} \end{pmatrix} \\ \underline{\underline{A}} - \underline{\underline{I}}\underline{\underline{I}} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - \begin{pmatrix} \underline{\underline{I}} & 0 \\ 0 & \underline{\underline{I}} \end{pmatrix} = \begin{pmatrix} A_{11} - \underline{\underline{I}} & A_{12} \\ A_{21} & A_{22} - \underline{\underline{I}} \end{pmatrix} \\ &\Rightarrow \\ |\underline{\underline{A}} - \underline{\underline{I}}\underline{\underline{I}}| &= \begin{vmatrix} A_{11} - \underline{\underline{I}} & A_{12} \\ A_{21} & A_{22} - \underline{\underline{I}} \end{vmatrix} \end{aligned}$$

And, as $|\underline{\underline{A}} - \underline{\underline{I}}\underline{\underline{I}}| = 0$:

$$\begin{vmatrix} A_{11} - \underline{\underline{I}} & A_{12} \\ A_{21} & A_{22} - \underline{\underline{I}} \end{vmatrix} = (A_{11} - \underline{\underline{I}})(A_{22} - \underline{\underline{I}}) - A_{12}A_{21} = 0$$

Hence, expanding out into a quadratic, we find the characteristic equation:

$$A_{11}A_{22} - I(A_{11} - A_{22}) + I^2 = 0$$

This gives a quadratic equation for I , which can be solved – this will usually give 2 eigenvalues – the roots to the quadratic.

Then, the eigenvectors can be found from:

The definition was $(\underline{\underline{A}} - I\underline{\underline{I}})\underline{\underline{e}} = 0$, so for each eigenvector e , there will be an eigenvalue I , so $(\underline{\underline{A}} - I_i\underline{\underline{I}})\underline{\underline{e}}_i = 0$

So, solving:

$$\begin{pmatrix} A_{11} - I & A_{12} \\ A_{21} & A_{22} - I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Will give two eigenvectors, one for each eigenvalue found.

The process is similar for 3x3, 4x4 matrices, but, due to the nature of the determinants, a cubic, or quartic, equation will come out – and they are generally hard to solve!

But examples are the best way to get your head round eigenvalues/vectors.