General Relativity, Cosmology & Dark Energy

Literature Review

Jonathan A. Pearson

Jodrell Bank Centre for Astrophysics
University of Manchester
jp@jb.man.ac.uk

February 17, 2010

Outline

1. General relativity: geometry, content & the big idea
2. The standard model of cosmology: FRW & inventing components
3. The nature of dark energy: scalar fields & domain walls

Notation Shorthand: $x^i = (x, y, z)$ or (r, θ, ϕ) "3-vector".

Include time, becomes 4-vector:

$$x^\mu = (t, x, y, z), \qquad \partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Geometry described using line element/metric: distance between two points living in a manifold – a set of points. Spacetime is an example of a manifold.

Simplest: Euclidean, Pythagoras' theorem

$$\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \sum_{i,j=1}^{3} \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j$$

Note that "metric", δ_{ij}, constant throughout space (and time).

More general version of Pythagoras' theorem:

$$\mathrm{d}s^2 = g_{\mu\nu}(x^\alpha)\mathrm{d}x^\mu \mathrm{d}x^\nu$$

Here, metric $g_{\mu\nu}(x^\alpha)$, changes with space-time location.

General Relativity: Geometry

Metric gives a notion of distance – without it, distance does not exist!

Create Einstein tensor $G_{\mu\nu}$ which contains information about geometry, via derivatives of metric

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \\ R &\sim \partial^2 g + \partial g \partial g \qquad \text{both very complicated functions of } g_{\mu\nu}. \end{aligned}$$

Example; $R = 2$ for "surface" of circle (i.e. it's curved!). Inside a sphere is not curved (it's just a bounded Euclidean space).

Satisfies "Bianchi identity"

$$\nabla_\mu G^{\mu\nu} = 0,$$

which is a conservation equation regarding geometry. All information about curvature is inside $G_{\mu\nu}$.

Dont be tempted to think about "bowling ball on trampoline" – extra dimension in which to curve is not required (and infact gets it wrong)

The **content** of a manifold is described via energy-momentum tensor $T_{\mu\nu}$:

$$T_{\mu\nu} = \text{energy density} \oplus \text{pressure}.$$

This also satisfies a conservation equation:

$$\nabla_{\mu} T^{\mu\nu} = 0.$$

Example: perfect fluid

$$T_{\mu\nu} = \text{diag}(\rho, p, p, p)$$

... link pressure p and energy density ρ via equation of state $w = p/\rho$.

Recall from undergrad days:

$$p_{\text{m}} = 0, \qquad p_{\gamma} = \frac{1}{3}\rho_{\gamma}.$$

Wanted: theory allowing us to understand how stuff moves in a curved manifold (curved meaning metric changes from place to place: distances change meaning from place to place – and time to time!).

We have: geometrical term $G_{\mu\nu}$ and content term $T_{\mu\nu}$ (both satisfying conservation equations).

So, equate them!

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \qquad \kappa = 8\pi G.$$

Geometry of a manifold is determined by distribution of stuff in the spacetime (and vice-versa)

Aside Can get Einsteins equation from minimising an action:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left(R - 2\kappa \mathcal{L}_{\rm mat}\right).$$

(remember from this morning: equations of motion that minimise action/energy)

Geometry The job is to write the distance measure of the universe, as a whole.

Remember Euclidean space uses Pythagoras' theorem. The model uses "Friedmann-Robertson-Walker" (FRW) line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right\}.$$

Expanding spatial part. Note $k = 0$ reduces to spherical polars (*inside* a sphere – flat).

Content Prescribe an energy-momentum tensor. Use

$$T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p) \qquad p = w\rho.$$

p is total pressure content, and ρ total energy density. Note that this models entire universe *as a fluid*!

Do some serious algebraic jiggery-pokery, and get Einsteins equations:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho\left(1+3w\right), \qquad H^2 = \frac{8\pi G}{3}\rho, \qquad \dot{\rho} + 3H\left(\rho + p\right) = 0.$$

Raychaudhuri equation, Friedmann equation & fluid equation.

Recall Raychaudhuri equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho\left(1 + 3w\right).$$

Acceleration corresponds to $\ddot{a} > 0$. This is achieved by having

$$w < -\frac{1}{3}.$$

No matter or radiation components have this equation of state ($w_{\mathrm{m}} = 0, w_{\mathrm{r}} = \frac{1}{3}$).

Invent something exotic that has $w_{?} < -\frac{1}{3}$, to "explain" the observed universal accelerated expansion. It must dominate the current energy content of the universe, so that it can accelerate.

Dark energy is born, but what is it?!

Know: must have $w_{\rm de} < -\frac{1}{3}$. Also must now be in dominance. Job is now to dream up models and ideas of things that have this property.

Scalar field: Write down Lagrangian and its energy-momentum tensor,

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi - V(\phi), \qquad T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\mathcal{L},$$

and find

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{T_{ii}}{T_{00}} = \frac{\dot{\phi}^2 - |\nabla\phi|^2 - 2V(\phi)}{\dot{\phi}^2 - |\nabla\phi|^2 + 2V(\phi)}.$$

Try setting $\dot{\phi} = 0$ (i.e. static field) and $|\nabla\phi| = 0$ (i.e. homogeneous field), and you get

$$w_\phi = -1.$$

Quintessence lets $\dot{\phi} \neq 0$, and still homogeneous. Thus field can evolve to have this equation of state! Can also change the kinetic term: K-essence.

How about a non-homogeneous non-static scalar field?

Domain walls Lagrangian, with specific potential

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2.$$

Solution to equation of motion:

$$\phi(x) = \eta \tanh\left(x\sqrt{\frac{\lambda\eta^2}{2}}\right) \qquad \text{“kink”}$$

Do a bit of algebra, and find

$$T_{\mu\nu} = \frac{1}{2}\lambda\eta^4 \operatorname{sech}^4\left(x\sqrt{\frac{\lambda\eta^2}{2}}\right) \operatorname{diag}(1,0,1,1).$$

Do even more, and find

$$w_{\text{dw}} = -\frac{2}{3} + v^2.$$

Therefore if $v = 0$ (the velocity of walls is zero), domain walls can be dark energy!

Called elastic dark energy model.

Summary

1. General relativity is a fusing of geometry and "stuff" content.
2. Standard model of cosmology
3. Need to invent dark energy to describe acceleration in FRW universe (how about a non-FRW universe? My next project!)
4. Static, homogeneous scalar fields do the job
5. As do domain walls! But need to make domain walls "freeze in", $v = 0$.

General Relativity, Cosmology & Dark Energy

Literature Review

Jonathan A. Pearson

Jodrell Bank Centre for Astrophysics
University of Manchester
jp@jb.man.ac.uk

February 17, 2010