

Probability Distributions:

P_i is the probability of outcome x_i .

$\langle x \rangle = \sum_i P_i x_i$ is the expectation value (or "average") = \bar{x}

Standard deviation = \mathbf{s} .

Variance:

$$\begin{aligned} \text{var}(x) = \mathbf{s}^2 &= \langle (x - \langle x \rangle)^2 \rangle \\ &= \langle x^2 - 2x\langle x \rangle + \langle x \rangle^2 \rangle \\ &= \sum_i P_i x_i^2 - 2\langle x \rangle \sum_i P_i x_i + \langle x \rangle^2 \underbrace{\sum_i P_i}_{=1} \\ &= \langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2 \\ &= \langle x^2 \rangle - \langle x \rangle^2 \\ \therefore \mathbf{s}^2 &= \langle x^2 \rangle - \langle x \rangle^2 \end{aligned}$$

Geometric Distributions:

Probability of success on "first go" = p .

So prob. of failure = $1 - p = q$.

$$P_n = pq^{n-1}$$

Need that this distribution adds to unity. i.e. that $\sum_{n=1}^{\infty} P_n = 1$:

$$\begin{aligned} \sum_{n=1}^{\infty} P_n &= p + pq + pq^2 + \dots \\ &= p + q(p + pq + pq^2 + \dots) \\ &= p + q \sum P_n \\ \therefore \sum P_n &= p + q \sum P_n \\ \Rightarrow \sum P_n - q \sum P_n &= p \\ \therefore \sum P_n (1 - q) &= p \\ \therefore \sum P_n &= \frac{p}{1 - q} \end{aligned}$$

But: $p = 1 - q$

$$\Rightarrow \sum_{n=1}^{\infty} P_n = \frac{p}{1 - q} = \frac{p}{p} = 1$$

See that the average now is given by $\langle n \rangle = \sum_{n=1}^{\infty} nP_n$. To compute this value:

$$\langle n \rangle = \sum_{n=1}^{\infty} nP_n = \sum npq^{n-1} = p \sum nq^{n-1}$$

Notice that:

$$nq^{n-1} = \frac{d}{dq}(q^n)$$

$$\Rightarrow \sum nq^{n-1} = \frac{d}{dq} \left(\sum q^n \right)$$

But, $\sum q^n$ is the sum of a geometric series.

So:

$$\begin{aligned} \sum nq^{n-1} &= \frac{d}{dq} \sum q^n = \frac{d}{dq} \left(\frac{q}{1-q} \right) \\ &= \frac{1}{1-q} + \frac{q}{(1-q)^2} \\ &= \frac{1}{(1-q)^2} \end{aligned}$$

$$\Rightarrow \langle n \rangle = p \sum nq^{n-1} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\therefore \langle n \rangle = \frac{1}{p}$$

To find the variance, calculate $\langle n^2 \rangle$, and substitute in. To do this: See that you need to find the expression for:

$$\begin{aligned} \sum n^2 P_n &= \sum n^2 pq^{n-1} \\ &= p \sum n^2 q^{n-1} \\ &= p \sum n \underbrace{nq^{n-1}}_{=\frac{d}{dq}(q^n)} \\ &= p \sum n \frac{d}{dq}(q^n) \\ &= p \frac{d}{dq} \sum nq^n \\ &= p \sum n(n-1)q^{n-1} \end{aligned}$$

Now, taking out q :

$$\therefore \sum n^2 P_n = pq \sum n(n-1)q^{n-2} = \sum (n^2 - n)P_n$$

And, see that:

$$\sum n(n-1)q^{n-2} = (-1) \left(-\frac{2}{(1-q)^3} \right)$$

$$\therefore pq \sum n(n-1)q^{n-2} = \frac{2pq}{(1-q)^3}$$

Hence:

$$\sum (n^2 - n)P_n = \frac{2pq}{(1-q)^3}$$

Notice that:

$$\sum (n^2 - n)P_n = \sum n^2 P_n - \sum nP_n = \langle n^2 \rangle - \langle n \rangle$$

Therefore, using the above expression for $\langle n \rangle$:

$$\langle n^2 \rangle = \langle n \rangle + \frac{2pq}{(1-q)^3} = \frac{1}{p} + \frac{2(1-p)}{p^2}$$

Hence, the variance $\text{var}(n)$ can be computed:

$$\text{var}(n) = \langle n^2 \rangle - \langle n \rangle^2$$

$$= \left(\frac{1}{p} + \frac{2(1-p)}{p^2} \right) - \left(\frac{1}{p} \right)^2$$

$$= \frac{1-p}{p^2}$$

Continuous Distributions:

Suppose a discrete prob.dist has a resolution down to Δx , so that $P_i = P(x_i < x < x_i + \Delta x)$.

Let $\Delta x \rightarrow 0$, so that:

$$P(x_i < x < x_i + \Delta x) = f(x_i)\Delta x$$

Hence:

$$P(a < x < b) = \lim_{\Delta x \rightarrow 0} \sum_i f(x_i)\Delta x = \int_a^b f(x)dx$$

With an expectation value $\langle x \rangle$:

$$\langle x \rangle = \int_{-\infty}^{+\infty} xf(x)dx$$

Notice that:

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

The variance and standard deviation is defined as it was before, but with the integral sign instead of a summation sign.

Gaussian Distributions:

Displayed as a bell-shaped graph. With the “peak” as the average m . The standard deviation s will show points a little to the right and left of m . They are usually given by functions of the form:

$$f(x) = \frac{e^{-(x-m)^2/2s^2}}{\sqrt{2ps^2}}$$

“Cumulative” functions $P(x)$ and $C(x)$ are to show probabilities of the function – for example, $P(m+s)$ gives the probability that a result lies *above* the $m+s$ value. And similarly $C(m-s)$ gives the probability below the point $m+s$. The main point of these is also that $P(m+s) = C(m-s)$; from the symmetry of the bell-curve.

Also, $P(m+2s) = C(m-2s)$. And so on.

These cumulative probabilities are defined by:

$$C(X) \equiv P(Y < X) = \int_{-\infty}^X f(y)dy$$

The P is not the above cumulative function $P(x)$; it is a probability function.

$P(x)$ can also be thought of as a survival probability:

$$P(X) \equiv 1 - C(X)$$

$$= \int_X^{+\infty} f(y)dy$$

An important integral result for Gaussian’s is below:

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \left(\frac{\pi}{a} \right)^{\frac{1}{2}}$$

An example of a Geometric Distribution:

Suppose you have an a particle, making F collisions per second, with a barrier it is trying to get out of. In a time t it has Ft attempts. Each attempt has probability p of success, so probability of failure = $(1 - p)$. So, the probability that the nucleus survives to time t is:

$$\begin{aligned} P(t) &= (1 - p)^{Ft} \\ &= \left(e^{\ln(1-p)} \right)^{Ft} \\ &= \left(e^{F \ln(1-p)} \right)^t \end{aligned}$$

Now, if you denote:

$$F \ln(1 - p) = -\frac{1}{T}$$

Then:

$$P(t) = e^{-\frac{t}{T}}$$

Now, if $f(t)$ is the probability distribution for decay times, then:

$$P(t) = \int_t^{\infty} f(t') dt'$$

So, on solving for $f(t)$:

$$f(t) = -\frac{dP}{dt} = \frac{1}{T} e^{-\frac{t}{T}}$$

An exponential probability distribution.

To check that this is a probability distribution, it should sum to unity. To do this:

$$\int_0^{\infty} f(t) dt = \frac{1}{T} \int_0^{\infty} e^{-\frac{t}{T}} dt$$

For later convenience, note that:

$$\begin{aligned} I(I) &\equiv \int_0^{\infty} e^{-It} dt \\ &= \left[-\frac{1}{I} e^{-It} \right]_0^{\infty} \\ &= \frac{1}{I} \end{aligned}$$

So, the original integral becomes:

$$\int_0^{\infty} f(t) dt = \frac{1}{T} I(1/T) = (1/T)/(1/T) = 1$$

An average and variance can also be calculated.

Poisson Distributions:

These are used when several events might occur independently of each other, and the mean rate of occurrence is known.

For example: Geiger counter clicks in time t , due to background radiation; Number of earthquakes in a given time; Number of molecules in a gas in a small volume.

To do this:

The probability of no event happening, in time t , in the presence of a constant hazard rate \mathbf{a} , which is its survival probability:

$$P_0(t; \mathbf{a}) = e^{-\mathbf{a}t}$$

Use this to obtain the probability of one event happening, $P_1(t; \mathbf{a})$, and then extend method to $P_2, P_3, \dots, P_k(t; \mathbf{a})$.

Note that $\mathbf{a}\Delta t$ is both the probability of one event happening, in a short period of time, and the mean number of events in a short time – as the probability of two events occurring in a small period of time is *very* small. So, \mathbf{a} = mean rate of occurrence.

So, to find $P_1(t; \mathbf{a})$:

Divide the time interval into 3 parts, assuming that the single event occurs in the short interval $[S, S + \Delta S]$. This has probability:

$$\Delta P_1 = P_0(S) \times \mathbf{a}\Delta S \times P_0(t - (S + \Delta S)) = P_0(S) \cdot \mathbf{a}\Delta S \cdot P_0(t - S)$$

Where:

$P_0(S)$ is the probability of no event upto S

$\mathbf{a}\Delta S$ is the probability of one event in ΔS

$P_0(t - S)$ is the probability of no events in remaining time $t - S$, as ΔS small

So:

$$\begin{aligned} \Delta P_1 &= e^{-\mathbf{a}S} \times \mathbf{a}\Delta S \times e^{-\mathbf{a}(t-S)} \\ &= e^{-\mathbf{a}t} \times \mathbf{a}\Delta S \end{aligned}$$

But, the event could occur in any interval ΔS , so:

$$P_1(t; \mathbf{a}) = \int_0^t e^{-\mathbf{a}t} \mathbf{a}dS = e^{-\mathbf{a}t} \mathbf{a}t$$

To do this for P_{k+1} :

Divide interval into 3 parts: P_k events in $0 \leq t \leq S$, one event in $S \leq t \leq S + \Delta S$, and no events in $S + \Delta S \leq t \leq T$. This has probability:

$$\Delta P_{k+1} = P_k(S) \times \mathbf{a}\Delta S \times P_0(t - S)$$

And, integrating as before:

$$P_{k+1}(t; \mathbf{a}) = \int_0^t P_k(S) P_0(t - S) \mathbf{a}dS$$

Now, when you start do calculate for different values of k , you will see that the general form is:

$$P_k(t; \mathbf{a}) = e^{-\mathbf{a}t} \frac{(\mathbf{a}t)^k}{k!}$$

Which is known as the Poisson Distribution.

Checking that this adds up to unity, writing $\mathbf{I} = \mathbf{a}t$:

$$\sum_{k=1}^{\infty} P_k = \sum e^{-\mathbf{I}} \frac{\mathbf{I}^k}{k!} = e^{-\mathbf{I}} \left\{ \sum_{k=0}^{\infty} \frac{\mathbf{I}^k}{k!} \right\} = e^{-\mathbf{I}} e^{\mathbf{I}} = 1$$

Making use of the series expansion for the exponential function.

The calculations can be done similarly for the expectation value:

$$\begin{aligned}
 \langle k \rangle &= \sum_{k=0}^{\infty} k P_k = \sum_{k=1}^{\infty} k \frac{\mathbf{I}^k}{k!} e^{-\mathbf{I}} \\
 &= \sum_{k=1}^{\infty} k \frac{\mathbf{I}^k}{k!} e^{-\mathbf{I}} \\
 &= \mathbf{I} e^{-\mathbf{I}} \left\{ \sum_{k=1}^{\infty} \frac{\mathbf{I}^{k-1}}{(k-1)!} \right\} \\
 &= \mathbf{I} e^{-\mathbf{I}} e^{\mathbf{I}} \\
 &= \mathbf{I}
 \end{aligned}$$

∴ $\langle k \rangle = \mathbf{I}$

Also, to find the variance, do similar operations, and one will find:

$$\text{var}(k) = \mathbf{s}_k^2 = \langle k^2 \rangle - \langle k \rangle^2 = \mathbf{I}$$

⇒ $\text{var}(k) = \langle k \rangle = \mathbf{I}$

For Poisson Distributions.

Sums of Poisson random variables:

Suppose l and m are Poisson random variables, with means \mathbf{I} and \mathbf{m} , respectively.

For example, they could be the numbers of hurricanes and tsunamis.

So, what's the distribution for their sum $s = \mathbf{I} + \mathbf{m}$?

In the example, s would be the total number of disasters – hurricanes + tsunamis.

See that s may be made up in various ways:

$$(l, m) : (s, 0), (s-1, 1), \dots, (s-k, k), \dots, (0, s)$$

All mutually exclusive possibilities.

So, the probability of scenario $l = s - k, m = k$ is:

$$\left\{ \frac{\mathbf{I}^{s-k}}{(s-k)!} e^{-\mathbf{I}} \right\} \times \left\{ \frac{\mathbf{m}^k}{k!} e^{-\mathbf{m}} \right\}$$

Which is to be summed over the possible values for k : $k = 0, 1, 2, \dots, s$:

$$\begin{aligned}
 P_s &= \left[\sum_{k=0}^s \frac{\mathbf{I}^{s-k}}{(s-k)!} \frac{\mathbf{m}^k}{k!} s! \right] \cdot \frac{e^{-(\mathbf{I}+\mathbf{m})}}{s!} \\
 &= (\mathbf{I} + \mathbf{m})^s \frac{e^{-(\mathbf{I}+\mathbf{m})}}{s!}
 \end{aligned}$$

Using the Binomial Theorem.

All this says, is that s is also a Poisson random variable, with mean $\langle s \rangle = \mathbf{I} + \mathbf{m}$

To extend this to n Poisson random variables l_1, l_2, \dots, l_n with means $\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_n$, then their sum:

$$s = \sum_i l_i$$

Is a Poisson random variable with mean:

$$\langle s \rangle = \sum_i \mathbf{I}_i$$

For non-constant hazard rate $\mathbf{a}(t)$, the Poisson distribution still holds:

The number of “counts” in the interval $[s, s + \Delta s]$ is a Poisson random variable, with

mean $\mathbf{a}(s)\Delta s$. Hence the total counts in $[0, t]$ is also a Poisson random variable, with mean:

$$\mathbf{I} = \sum_{\text{all_intervals}} \mathbf{a}(s)\Delta s \rightarrow \int_0^t \mathbf{a}(s)ds$$

For large I , Poisson distributions resemble Gaussian's. To do this, we need to look at:

$$P_n = \frac{I^n}{n!} e^{-I}$$

Near its maximum, as a function of n . That is, within a few standard deviations \sqrt{I} of the mean I . To do this properly, we need an analytical approximation to $n!$. This is done using Stirling's Approximation:

Consider the integral:

$$\begin{aligned} I &= \int_1^n \ln x dx \\ &= \int_1^n 1 \ln x dx \\ &= [x \ln x]_1^n - \int_1^n x \cdot \frac{1}{x} dx \\ &= n \ln n - n + 1 \end{aligned}$$

Now, approximate I by a series of rectangles *over* the $y = \ln x$ graph. For rectangles that terminate above the curve:

$$I < \ln 2 + \ln 3 + \dots + \ln n = \ln(n!)$$

And, for rectangles with terminate just below the curve:

$$I > \ln 1 + \ln 2 + \dots + \ln(n-1)$$

Now, notice:

$$\begin{aligned} \ln 1 + \ln 2 + \dots + \ln(n-1) &= \ln 1 + \ln 2 + \dots + \ln(n-1) + \ln n - \ln n \\ &= \ln(n!) - \ln n \end{aligned}$$

Hence:

$$I > \ln(n!) - \ln n$$

Therefore:

$$\begin{cases} \ln(n!) > I = n \ln n - n + 1 \\ \ln(n!) < I + \ln n = n \ln n - n + 1 + \ln n \end{cases}$$

Giving lower and upper bounds on $\ln(n!)$. To get an approximation to $\ln(n!)$, average the two bounds – add up, and divide by 2:

$$\ln(n!) \approx n \ln n - n + 1 + \ln(\sqrt{n})$$

Another argument changes the +1 to $\ln(\sqrt{2p}) = 0.919$, giving Stirling's Approximation, which is:

$$\ln(n!) \approx n \ln n - n + \ln(\sqrt{2pn})$$

The approximation improves with increasing n .

When this approximation is inserted into an expression for a Poisson distribution, use $f(n) = \ln P_n$. So:

$$\begin{aligned} f(n) &= \ln \left\{ \frac{I^n}{n!} e^{-I} \right\} \\ &= n \ln I - I - \ln(n!) \\ &= n \ln I - I - \left\{ n \ln n - n + \ln(\sqrt{2pn}) + \frac{1}{2} \ln n \right\} \end{aligned}$$

Now, the max occurs at $n = \hat{n}$, with $f'(\hat{n}) = 0$:

$$f'(n) = \ln I - \ln n - \frac{1}{2n}$$

For $I \gg 1 \Rightarrow \hat{n} \sim I$, as a first approximation, as the correction $-\frac{1}{2\hat{n}} = -\frac{1}{2I}$ is small compared with other terms.

Now, doing a Taylor expansion of $f(n)$ about $n = \hat{n}$:

$$f(n) = f(\hat{n}) + \frac{(n - \hat{n})}{1!} f'(\hat{n}) + \frac{(n - \hat{n})^2}{2!} f''(\hat{n}) + \dots$$

But, the first derivative is zero, as a maximum, so, for a first approximation:

$$f(\hat{n}) = f(I) = -\ln(\sqrt{2pI})$$

And:

$$\begin{aligned} f''(\hat{n}) &= \frac{d}{dn} \left\{ \ln I - \ln n - \frac{1}{2n} \right\} \Bigg|_{n=I} \\ &= \left\{ -\frac{1}{n} + \frac{1}{2n^2} \right\} \Bigg|_{n=I} \\ &= -\frac{1}{I} \end{aligned}$$

Hence, putting the two together:

$$f(n) \approx -\ln(\sqrt{2pI}) - \frac{(n - I)^2}{2I}$$

Remembering the initial function:

$$f(n) = \ln P_n$$

Hence:

$$P_n = \frac{e^{-\frac{(n-I)^2}{2I}}}{\sqrt{(2pI)}}$$

Which is the *Gaussian Limit of the Poisson Distribution*.

Binomial Distributions:

Consider an experiment of n independent trials, each having two possible outcomes: success or failure.

Probability of success p .

Probability of failure $q = 1-p$.

One possibility is that we have k successes, followed by $(n - k)$ failures, with probability $\underbrace{p \times p \times \dots \times p}_{k_times} \times \underbrace{q \times q \times \dots \times q}_{n-k_times}$

But, the successes and failures could occur in any one of:

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

So, the total probability of k successes will be:

$$P_k = \binom{n}{k} p^k q^{n-k}$$

Which is known as the Binomial Distribution.

To check that this is a probability distribution, it needs to add to unity; so require that:

$$\sum_{k=0}^n P_k = \sum_{k=0}^n \binom{n}{k} p^k q^{n-k} = 1$$

Now, the Binomial distribution is actually from the binomial expansion, so:

$$\sum_{k=0}^n \binom{n}{k} p^k q^{n-k} = (p + q)^n$$

But, $p + q = 1$, so:

$$\sum_{k=0}^n P_k = \sum_{k=0}^n \binom{n}{k} p^k q^{n-k} = (p + q)^n = 1$$

As required.

To find the expected number of successes $\langle k \rangle$:

$$\begin{aligned} \langle k \rangle &= \sum_{k=0}^n k P_k \\ &= \sum_{k=0}^n k \binom{n}{k} p^k q^{n-k} \\ &= q^n \sum_{k=0}^n \binom{n}{k} k X^k \end{aligned}$$

Where:

$$X = \frac{p}{q}$$

Notice that:

$$kX^k = X \frac{d}{dX} (X^k)$$

So:

$$\langle k \rangle = q^n X \frac{d}{dX} \sum_{k=0}^n \binom{n}{k} X^k 1^{n-k}$$

$$\begin{aligned}
 &= q^n X \frac{d}{dX} (X+1)^n \\
 &= q^n X n (X+1)^{n-1} \\
 &= q^n \frac{p}{q} n \left(\frac{p}{q} + 1 \right)^{n-1} \\
 &= q^{n-1} p n q^{n+1} (p+q)^{n-1} \\
 &= np
 \end{aligned}$$

So:

$$\langle k \rangle = np$$

For the Binomial Distribution.

To find the variance, use the same approach as in the Poisson:

$$\begin{aligned}
 \langle k(k-1) \rangle &= \langle k^2 \rangle - \langle k \rangle \\
 &= [\mathbf{s}_k^2 + \langle k \rangle^2] - \langle k \rangle
 \end{aligned}$$

So:

$$\begin{aligned}
 \langle k(k-1) \rangle &= \sum_{k=0}^n k(k-1) \binom{n}{k} p^k q^{n-k} \\
 &= q^n \sum_{k=0}^n \binom{n}{k} \underbrace{k(k-1) X^k}_{= X^2 \frac{d^2}{dX^2} X^k} \\
 &= q^n X^2 \frac{d^2}{dX^2} \sum_{k=0}^n \binom{n}{k} X^k \\
 &= q^n X^2 \frac{d^2}{dX^2} (X+1)^n \\
 &= q^n X^2 n(n-1) (X+1)^{n-2} \\
 &= n(n-1) p^2
 \end{aligned}$$

Hence:

$$\begin{aligned}
 \mathbf{s}_k^2 &= n(n-1) p^2 - \langle k \rangle^2 + \langle k \rangle \\
 &= (n^2 p^2 - np^2) - n^2 p^2 + np \\
 &= n(p - p^2) = npq
 \end{aligned}$$

So:

$$\mathbf{s}_k^2 = npq$$

Infact, for large n , the Binomial Distribution can be approximated by a Gaussian Distribution, similarly to the Poisson previously. To show this:

$$P_k = \binom{n}{k} p^k q^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

And:

$$\langle k \rangle = \bar{k} = np$$

So:

$$p = \frac{\bar{k}}{n}$$

Hence, rewriting P_k :

$$P_k = \frac{n!}{k!(n-k)!} \left(\frac{\bar{k}}{n}\right)^k \left(1 - \frac{\bar{k}}{n}\right)^{n-k}$$

Now:

$$\frac{n!}{k!(n-k)!} = n(n-1)\dots(n-k+1) = n^k \underbrace{\left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)}_{\substack{\rightarrow 1 \\ \text{for} \\ n \rightarrow \infty}}$$

Also:

$$\left(1 - \frac{\bar{k}}{n}\right)^n \rightarrow e^{-\bar{k}}$$

Hence, P_k becomes:

$$P_k \approx \frac{n^k}{k!} \left(\frac{\bar{k}/n}{1 - \bar{k}/n}\right)^k e^{-\bar{k}} \rightarrow \frac{\bar{k}^k}{k!} e^{-\bar{k}}$$

Which is the Poisson Distribution.