

Moment of Inertia Tensor:

$$I_{ij} = m_a (r_a^2 \mathbf{d}_{ij} - r_{a,i} r_{a,j})$$

Where:  $\underline{r}_a = r_{a,1} \hat{e}_1 + r_{a,2} \hat{e}_2 + r_{a,3} \hat{e}_3$

Such that:  $r_a^2 = r_{a,1}^2 + r_{a,2}^2 + r_{a,3}^2$

Making use of:  $\mathbf{r} = \frac{m}{V} = \frac{dm}{dV} \Rightarrow dm = \mathbf{r} dV$

For a body with  $\mathbf{a}$  -elements.

This gives the elements of the tensor:

$$\underline{\underline{I}} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

The evaluation of the elements can be simplified in a number of ways:

Perpendicular axis theorem:  $I_{11} + I_{22} = I_{33}$

To derive this:

Take a 2D object, with surface mass-density  $\mathbf{s}(\underline{r})$ , in the  $x_1, x_2$  plane.

So:  $I_{ij} = \int_A \mathbf{s}(\underline{r}) (r^2 \mathbf{d}_{ij} - r_i r_j) dS$

Choose  $x_3 = 0$  for all points in the plane – symmetry axis.

This quickly gives:  $I_{13} = I_{31} = 0$  &  $I_{23} = I_{32} = 0$

Now:

$$I_{11} = \int_A \mathbf{s}(\underline{r}) (r_1^2 + r_2^2 - r_1^2) dS = \int_A \mathbf{s}(\underline{r}) r_2^2 dS$$

$$I_{22} = \int_A \mathbf{s}(\underline{r}) r_1^2 dS$$

$$\Rightarrow I_{33} = I_{11} + I_{22}$$

$$I_{33} = \int_A \mathbf{s}(\underline{r}) (r_1^2 + r_2^2) dS$$

Now, the principle axes of inertia have the property that  $\underline{L}$  &  $\underline{\omega}$  are parallel for

rotations about these principle axes. Suppose that  $\underline{e}_i$  is a principle axis. Then:

$$\underline{\underline{I}} \underline{e} = I \underline{e}, \text{ showing that } \underline{e} \text{ is an eigenvector, with } I \text{ an eigenvalue.}$$

This will give a diagonal tensor, with elements the ‘principle moments of inertia’. The principle axes of a body are also its axes of symmetry.

$\underline{L} = \underline{\underline{I}} \cdot \underline{\omega}$	Angular momentum
$T = \frac{1}{2} \underline{\omega} \cdot \underline{L} = \frac{1}{2} \underline{\omega} \cdot (\underline{\underline{I}} \cdot \underline{\omega})$	Kinetic energy of rotation
$\underline{\mathbf{t}} = \underline{\omega} \times \underline{L}$	Torque

Eulers Equations:

We have:  $\underline{\mathbf{t}} = \left( \frac{d\underline{\mathbf{L}}}{dt} \right)_{rot} + \underline{\mathbf{w}} \times \underline{\mathbf{L}}$  (see non-inertial reference frame stuff)

Eventually giving:

$$\underline{\mathbf{t}} = \underline{\mathbf{I}}\underline{\dot{\mathbf{w}}} + \underline{\mathbf{w}} \times (\underline{\mathbf{I}}\underline{\mathbf{w}})$$

Which, after evaluating all the cross-products gives Euler's equations:

$$\begin{aligned} \mathbf{t}_1 &= I_1 \dot{\mathbf{w}}_1 + (I_3 - I_2) \mathbf{w}_2 \mathbf{w}_3 \\ \mathbf{t}_2 &= I_2 \dot{\mathbf{w}}_2 + (I_1 - I_3) \mathbf{w}_1 \mathbf{w}_3 \\ \mathbf{t}_3 &= I_3 \dot{\mathbf{w}}_3 + (I_2 - I_1) \mathbf{w}_1 \mathbf{w}_2 \end{aligned}$$

These can be applied:

Suppose  $\underline{\mathbf{t}} = 0$ , and the principle axes of inertia are such that  $I_1 = I_2 = I$ . It can then be easily shown that  $\dot{\mathbf{w}}_3 = 0$ . Then you are left with:

$$\begin{cases} I\dot{\mathbf{w}}_1 + (I_3 - I)\mathbf{w}_2 \mathbf{w}_3 = 0 \\ I\dot{\mathbf{w}}_2 + (I - I_3)\mathbf{w}_1 \mathbf{w}_3 = 0 \end{cases}$$

If you define  $\Gamma = \left( \frac{I_3 - I}{I} \right) \mathbf{w}_3$ , then the above equations simplify easily to:

$$\begin{cases} \dot{\mathbf{w}}_1 + \Gamma \mathbf{w}_2 = 0 \\ \dot{\mathbf{w}}_2 - \Gamma \mathbf{w}_1 = 0 \end{cases} \Rightarrow \begin{cases} \dot{\mathbf{w}}_1 = -\Gamma \mathbf{w}_2 \\ \dot{\mathbf{w}}_2 = \Gamma \mathbf{w}_1 \end{cases}$$

So, differentiating:

$$\begin{cases} \ddot{\mathbf{w}}_1 + \Gamma \dot{\mathbf{w}}_2 = 0 \\ \ddot{\mathbf{w}}_2 - \Gamma \dot{\mathbf{w}}_1 = 0 \end{cases}$$

Inserting the above expressions for  $\dot{\mathbf{w}}_1, \dot{\mathbf{w}}_2$ :

$$\begin{cases} \ddot{\mathbf{w}}_1 + \Gamma^2 \mathbf{w}_1 = 0 \\ \ddot{\mathbf{w}}_2 + \Gamma^2 \mathbf{w}_2 = 0 \end{cases}$$

Which are just expressions for SHM, assuming that  $\Gamma^2 > 1$  (otherwise get 'complex' motion involving hyperbolics!).

So:  $\mathbf{w}_1 = A \cos(\Gamma t + \mathbf{f})$  as  $\dot{\mathbf{w}}_1 = -\Gamma \mathbf{w}_2$   
 $\mathbf{w}_2 = A \sin(\Gamma t + \mathbf{f})$

With appropriate constants for amplitude & phase difference.

Gravitation:

$$\underline{F} = -\frac{GMm}{r^2} \hat{e} \quad (\hat{e} \text{ here is a unit vector in the opposite direction to the central body})$$

Keplers laws of planetary motion:

- 1) Law of ellipses: Orbit of each planet is an ellipse, with the Sun at one foci.
- 2) Law of equal areas: A line drawn between the Sun and a planet sweeps out an equal area in a equal time.
- 3) Harmonic law:  $T^2 \propto R^3$

Conservative forces: work done by the force is independent of the path.

For conservative forces *only*, there is a potential energy function  $U(\underline{x})$ , such that:

$\underline{F} \cdot d\underline{x} = -dU$ , from which you can show:

$$\underline{F} = -\nabla U$$

Reduced mass:  $\underline{m} = \frac{m_1 m_2}{m_1 + m_2}$

And, Newton:  $\underline{m}\ddot{\underline{r}} = \underline{f}(\underline{r})$

Conservation of angular momentum:  $\frac{dA}{dt} = \frac{L}{2m}$

$$r = \frac{a}{1 + e \cos \varphi} \quad e = \sqrt{1 + \frac{2hI^2}{G^2 M^2}} \quad I = \frac{L}{m} \quad \mathbf{h} = \frac{E}{m}$$

$$a = \frac{I^2}{GM} \quad T = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

Total energy:  $-\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r} = E$

If you want to work out the period of a circular orbit, say, remember

that  $F = mr\omega^2$  and that  $\omega = 2\pi f = \frac{2\pi}{T}$  and that  $F = \frac{GMm}{r^2}$ , then rearrange.

Non-Inertial Reference Frames:

For any position vector  $\underline{W}$  in a rotating reference frame:

$$\frac{d\underline{W}}{dt} = \left. \frac{d\underline{W}}{dt} \right|_{rot} + \underline{\omega} \times \underline{W}$$

For example, if a position vector  $\underline{r}$ , then the velocity vector  $\underline{v}$  is given by:

$$\underline{v} = \frac{d\underline{r}}{dt} = \left. \frac{d\underline{r}}{dt} \right|_{rot} + \underline{\omega} \times \underline{r} \Rightarrow \underline{v} = \underline{v}' + \underline{\omega} \times \underline{r}$$

Do it again for acceleration:

$$\underline{a} = \frac{d\underline{v}}{dt} = \left. \frac{d}{dt} (\underline{v}' + \underline{\omega} \times \underline{r}) \right|_{rot} + \underline{\omega} \times (\underline{v}' + \underline{\omega} \times \underline{r}) \quad (\text{Assume } \dot{\underline{\omega}} = 0)$$

So:

$$\underline{a} = \underline{a}' + 2\underline{\omega} \times \underline{v}' + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

And, from  $\underline{F} = m\underline{a}$ :

$$\underline{F} - 2m\underline{\omega} \times \underline{v}' - m\underline{\omega} \times (\underline{\omega} \times \underline{r}) = m\underline{a}'$$

So:

Coriolis force is the  $2m\underline{\omega} \times \underline{v}'$  part.

Centrifugal force is the  $m\underline{\omega} \times (\underline{\omega} \times \underline{r})$  part.

If you define:

$\underline{e}_1$  as North

$\underline{e}_2$  as Up

$\underline{e}_3$  as East

$I$  as latitude

Then, if:  $\underline{x} = 0\underline{e}_1 + 0\underline{e}_2 + R\underline{e}_3$

$\underline{\omega} = (\cos I \underline{e}_2 + \sin I \underline{e}_3)\underline{\omega} = \underline{\omega}(0, \cos I, \sin I)$

Then all the above forces can be found by evaluating the cross products.

Relativity:

Lorentz transformations:

$$x'_1 = \mathbf{g}(v)(x - vt)$$

$$t' = \mathbf{g}(v)\left(t - \frac{vx}{c^2}\right) \qquad \mathbf{g}(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Velocity transformations:

$$V_x = \frac{dx}{dt} = \frac{V'_x + v}{1 + \frac{vV'_x}{c^2}} \qquad V_y = \frac{dy}{dt} = \frac{V'_y}{\mathbf{g}\left(1 + \frac{vV'_x}{c^2}\right)}$$

$$E = \mathbf{g}(v)mc^2 \quad \underline{p} = \mathbf{g}(v)m\underline{v} \quad \text{both imply that:} \quad E^2 = p^2c^2 + m^2c^4$$

Define  $\underline{x} = (\underline{x}, ict)$  a 4-vector.

Now, this is true:  $\underline{x} \cdot \underline{x} = \underline{x}' \cdot \underline{x}'$  -invariance

Now, in 3D:  $x'_i = R_{ij}x_j$

So, in the 4D version:  $x'_m = L_{mn}x_n$  so  $\underline{x}' = \underline{L}\underline{x}$

You can think of  $\underline{L}$  as a 4D 'rotation' matrix.

So:  $\underline{x} \cdot \underline{x} = x_m x_m = \underline{x}^2 - c^2 t^2 = -c^2 \mathbf{t}^2$

Where we have defined the proper time:

$$\mathbf{t}^2 = (t_1 - t_2)^2 - \frac{1}{c^2}(\underline{x}_1 - \underline{x}_2)^2$$

We can define a 4-velocity:  $\underline{u} = \frac{d\underline{x}}{dt} = \mathbf{g}(u)(\underline{u}, ic)$

Therefore:

$$\underline{L} = \begin{pmatrix} \mathbf{g} & 0 & 0 & \frac{i\mathbf{g}v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{i\mathbf{g}v}{c} & 0 & 0 & \mathbf{g} \end{pmatrix}$$

Which can be used to derive the velocity transformations.

The wave 4-vector:  $\underline{k} = \left(k, \frac{i\omega}{c}\right)$

Momentum 4-vector:  $\underline{p} = \mathbf{g}(v)(\underline{p}, imc)$

$$\underline{p} \cdot \underline{p} = -m^2c^2$$