No damping, no force:

$$\ddot{x} = -w^2 x \qquad w = \sqrt{\frac{k}{m}}$$

$\Rightarrow$

$$x = A\sin(wt + f) = Ae^{i(wt+f)}$$

Damping force *b*. no forcing term:

$$m\ddot{x} + b\dot{x} + kx = 0$$

Solution: $x = Ae^{\left(iw - \frac{g}{2}\right)t}$ $\qquad g = \frac{b}{m}$

$\Rightarrow$

$$w = w_0 - \frac{g^2}{4} \qquad \left(w_0 = \sqrt{\frac{k}{m}}\right)$$

So, the damping term is $e^{-\frac{gt}{2}} = e^{-\frac{bt}{2m}}$, so bigger b $\Rightarrow$ faster damping

$Q = \frac{w_0}{g}$ Quality factor of the motion

So: $w = w_0\sqrt{1 - \frac{1}{4Q^2}}$

Forced & damped:

$$m\ddot{x} + b\dot{x} + kx = Fe^{iwt}$$

As before: $w_0 = \sqrt{\frac{k}{m}}$ $\quad w = w_0 - \frac{g^2}{4}$ $\quad g = \frac{b}{m}$ $\quad Q = \frac{w_0}{g}$

Solution: $x = A'e^{iwt}$

Hence, the equation of motion can be written as:

$-mw^2A' + iwbA' + kA' = F$

Where $A' = \frac{F}{-mw^2 + k + iwb}$ - which is a 'response' amplitude/phase

Alternatively: $A' = \frac{F/m}{w_0^2 - w^2 + igw}$

Taking the real part:

$$A = \frac{F/m}{\sqrt{\left(w^2 - w_0^2\right)^2 + g^2w^2}} \qquad \text{(amplitude)}$$

It appears that: $A' = Ae^{id}$, with $\tan d = \frac{gw}{w_0^2 - w^2}$ (phase)

$$w_{\max} = w_0\sqrt{1 - \frac{1}{2Q^2}}$$

LC-Circuits:

Inductor	L	$Z = j?L$	$V = L\frac{dI}{dt} = L\dot{I}$	
Charge	Q			
Resistance	R	$Z = R$	$V = IR$	(Z = impedance)
Voltage	V			
Capacitor	C	$Z = 1/j?C$	$V = Q/C$	

Voltage across capacitor: $V_C = \frac{Q}{C}$

Voltage across inductor; $V_L = L\dot{I}$

So, for (just) an LC circuit:

$$L\dot{I} + \frac{Q}{C} = 0 \qquad \dot{Q} = I$$

$$\Rightarrow$$

$$L\ddot{I} + \frac{1}{C}I = 0 \qquad w = \frac{1}{\sqrt{LC}}$$

So, the solution is: $I = I_0 \sin(wt + f)$

Add a resistor R (like the damping term before):

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0$$

$$\rightarrow \qquad \text{with:} \quad w_0 = \frac{1}{\sqrt{LC}} \text{ and } g = \frac{R}{L}$$

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}I = 0$$

So, the quality term: $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$ (Q is NOT the charge above!!!)

Add a voltage source $V(t) = Ve^{jwt}$ (here, $j = \sqrt{-1}$)

Solution: $I(t) = Ie^{jwt}$

Therefore:

$$-w^2 LI + jwRI + \frac{I}{C} = jwV$$

Giving:

$$I = \frac{jwV}{1/C - w^2 L + jwR} = \frac{jwV/L}{w_0^2 - w^2 + jwR/L}$$

Beats & coupled SHM:

$$\cos(w_1 t)+\cos(w_2 t)=2\cos\left(\frac{w_1+w_2}{2}t\right)\cos\left(\frac{w_1-w_2}{2}t\right)$$

From a trig identity

The second term modulates the beat, and the first provides the 'frequency' of the beat.

Coupled, same mass m:

Coupled pendulums, by a spring

$$m\ddot{x}_1=-\frac{mg}{l}x_1-k(x_1-x_2)$$

$$m\ddot{x}_2=-\frac{mg}{l}x_2-k(x_2-x_1)$$

Adding gives:

$$m(\ddot{x}_1+\ddot{x}_2)=-\frac{mg}{l}(x_1+x_2)$$

Say $X=x_1+x_2$, then:

$$\ddot{X}=-w_1^2X$$ which is of the usual form for SHM

With $w_1=\sqrt{\frac{g}{l}}$, giving a solution of:

$$X=A_1\cos(w_1t+f_1)$$ [or in whatever form]

Subtracting initial equations gives:

$$m(\ddot{x}_1-\ddot{x}_2)=-\frac{mg}{l}(x_1-x_2)-2k(x_1-x_2)$$

And, denoting $x=x_1-x_2$, gives a solution of:

$$x=A_2\cos(w_2t+f_2)$$ [or in whatever form]

With $w_2^2=\frac{g}{l}+2\frac{k}{m}$.

X and x are the normal modes of the system – n modes in a system of n coupled bodies.

Rewrite the equations as: $x_1=\frac{1}{2}(X+x)$ and $x_2=\frac{1}{2}(X-x)$, so that:

$x_1(t)=\frac{A}{2}(\cos(w_1t)+\cos(w_2t))$ (if the amplitudes and phases are equal)

So, using the function above for beats, the two expressions can be written as:

$$x_1(t)=A\cos\left(\frac{w_1+w_2}{2}t\right)\cos\left(\frac{w_1-w_2}{2}t\right)$$

$$x_2(t)=A\sin\left(\frac{w_1+w_2}{2}t\right)\sin\left(\frac{w_1-w_2}{2}t\right)$$

Spring with two masses on either end:

$$m\ddot{x}_1=-k(x_1-x_2)$$

$$m\ddot{x}_2=-k(x_2-x_1)$$

Then the normal modes are:

$x = x_1 - x_2$ gives $m\ddot{x} = -2kx$, giving an angular frequency $w = \sqrt{\frac{2k}{m}}$, with solution of the form $x = A\cos(wt + f)$

The other mode, $\ddot{X} = 0$, gives an un-interesting oscillator system: the centre of gravity proceeds with constant velocity.

Fourier transforms:

For odd, periodic functions:

$$f(q) = \sum_n a_n \sin(nq) \qquad \text{with:}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(q)\sin(nq)dq$$

For even, periodic functions:

$$f(q) = \sum_n b_n \cos(nq) \qquad \text{with:}$$

$$b_n = \frac{1}{(1+d_{n0})p}\int_{-p}^{p} f(q)\cos(nq)dq$$

Generally however, one can express a function as the sum of an even & odd functions:

$$f(q) = \sum_n a_n \sin(nq) + b_n \cos(nq)$$

Waves:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Where $v = \sqrt{\frac{T}{m}}$, with m (kg/m) the mass density of a string with some tension T.

- This is for transverse 1-D waves
- Derived using $F = ma$ and tensions on either side of an element.

For longitudinal waves:
e.g. spring, or a metal bar compressed at one end.
Gives:

$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$, but with $v = \sqrt{\frac{k}{m}}$, for a spring constant k.

- derived using Young's Modulus, and strain/stress

Solutions to wave-equations:
Any $f(x - vt)$ or $f(x + vt)$ is a solution.
The solution $f(x - vt)$ represents a wave travelling in the $+x$ direction, and the other solution is $-x$ travel.
So an example solution would be: $f(x,t) = A\sin(k(x \pm vt)) = A\sin(kx \pm wt)$,
remembering that $w = kv$.
Here, k is the wave number $= \frac{2p}{l}$, but as usual, $w = 2pf$.

Hence, wave motion can be written in any of the following forms:
$\sin(kx - wt)$ $\quad\cos(kx - wt)$ $\quad e^{i(kx-wt)}$ $\quad\ldots$
Speed of wave given by whatever multiplies x over whatever multiplies t.
$v = {}^{w}\!/_{k} = {}^{l}\!/_{T} = fl$, and the speed of a wave on a string is $v = \sqrt{{}^{T}\!/_{m}}$.

Waves at boundaries:

A wave is incident on a boundary – with different mediums:

$Ie^{i(wt-kx)}$ & $v = \sqrt{\frac{T}{m}}$ Before

$Te^{i(wt-k'x)}$ & $v' = \sqrt{\frac{T}{m'}}$ After

Note the same w and T. These are part of the 'boundary conditions'.
The refractive index is given by:

$$n = \frac{v}{v'} = \frac{l}{l'} = \frac{k'}{k}$$

Another boundary condition is that the join must be smooth.
With just the two waves, Incident & Transmitted, this is not satisfied, so introduce Reflected wave, hence:
$Ie^{i(wt-kx)} + R.e^{i(wt+kx)} = Te^{i(wt-k'x)}$,

Evaluating $\frac{\partial}{\partial x}$ on both sides give:

$-ikIe^{i(wt-kx)} + ikR.e^{i(wt+kx)} = -ik'Te^{i(wt-k'x)}$

Cancelling, and using *x = 0* at the boundary;

I + R = T -*ikI + ikR = ik'T*

Gives some relationships:

$\frac{T}{I} = \frac{2k}{k+k'} = \frac{2}{1+n}$ $\frac{R}{I} = \frac{k-k'}{k+k'} = \frac{1-n}{1+n}$

If *k* and *k'* are similar, then *R* is small – as expected.

Standing waves:

Consider a very heavy second rope: *v'* small *k'* large, so *T -> 0*, and *R -> -1*.

So the wave on the left:

$Ie^{i(wt-kx)} - Ie^{i(wt+kx)} = Ie^{iwt}\left(e^{-ikx} - e^{ikx}\right) = f_t(t)\,f_x(x)$, which, from the function:

$\sin q = \frac{e^{iq} - e^{-iq}}{2i}$, one can see that you get $2\sin(wt)\sin(kx)$ out.

Nodes (no disturbance) and anti-nodes (max disturbance) form every half wavelength

$\sin(kLx) = 0$, with $kL = p, 2p, 3p\ldots$ (divided by wavelength).

Only certain wavelengths are allowed, and:

$w = \frac{Nvp}{L}$ $f = \frac{Nv}{2L}$

Speed of sound:

Use fact that $pV^g = const$ hence, $p_0V_0^g = pV^g$

So, $p = \frac{p_0V_0^g}{V^g}$, therefore: $\frac{dp}{dV} = -\frac{gp_0V_0^g}{V^{g+1}} = -\frac{gp_0V_0^g}{V.V^g}$

As $p = \frac{p_0V_0^g}{V^g}$, then $\frac{dp}{dV} = -\frac{gp}{V} \Rightarrow dp = -\frac{gp}{V}dV$

Now, $p = \frac{F}{A}$ and $V = lA$ (volume = length x area), so $dV = A.dl$

Hence, from all the above:

$\frac{F}{A} = -\frac{gpdV}{V} = -\frac{gpdlA}{lA}$

$\therefore F = -\frac{gpdlA}{l}$ Calling $K = gpA$

And, as $v = \sqrt{\frac{K}{m}} = \sqrt{\frac{gpA}{m}}$ (remember that $r = m/V$ and $m = m/A$)

Therefore: $v = \sqrt{\frac{gp}{r}}$

The $g = \frac{n+2}{n}$, with *n* the number of degrees of freedom of molecule, so for diatomic air, $g = \frac{5+2}{5} = 1.4$

The assumptions used in this derivation are:

$V \propto 1/p$ Constant temperature

Not isothermal changes;

Is adiabatic (no energy flows in or out…);

Air is diatomic.

Standing waves:

Strings, with each end fixed…

$$L = Nl/2$$

Pressure: Displacement:

Pressure node Displacement node (D-node)

Pressure anti-node Displacement anti-node

Pipes:

Closed-closed

D-nodes

$L = Nl/2$

Open-open

D-anti-nodes

$L = Nl/2$

Closed-open

D-node @ closed end, anti-node @ open

$L = l/4$

Energy & power:

For a mass on a spring; $x(t) = a\sin wt$
It's KE= $\frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}ma^2w^2\cos^2 wt$

And to find its PE, remember that $F = -kx$:
$PE = -\int F dx \Rightarrow PE = \frac{1}{2}kx^2 = \frac{1}{2}ka^2\sin^2 wt$
To find total energy = PE + KE, first remember that $w^2 = \frac{k}{m} \Rightarrow mw^2 = k$
So:
$PE + KE = \frac{1}{2}ka^2(\sin^2 wt + \cos^2 wt)$
$= \frac{1}{2}ka^2$

Total energy per unit length, of a *transverse wave*, is given by $E = \frac{1}{2}Tk^2a^2$ (its an average).

Actually given by: $T\left(\frac{\partial f}{\partial x}\right)^2$; per unit length.

Remember that $w^2 = k^2v^2$ and $v^2 = T/m$.

Total energy per unit length of a *longitudinal wave*, is given by $\frac{1}{2}K\left(\frac{\partial f}{\partial x}\right)^2$.

Energy/pressure in sound waves…

If a displacement $= a\sin(kx - wt)$, then the strain $= \frac{\partial f}{\partial x} = ka\cos(kx - wt)$.
And, force = K x *strain* = $Kka\cos(kx - wt)$.
Also, $K = gp_0A$, so pressure = force/area = $gp_0ka\cos(kx - wt)$.
Inserting expressions for k, and things…

> Pressure amplitude of oscillation = $p = \frac{2pgp_0 fa}{v}$

With: a = amplitude, f = frequency, p_0 = atmospheric pressure…

Kinetic energy density is $\frac{1}{2}rv^2 = \frac{1}{2}rw^2a^2\cos^2(kx - wt)$, as the cosine term gives ½ as an average, and the PE density is equal:

> Total average energy density $= \frac{1}{2}rw^2a^2$

For a wave traversing (velocity v, in time t) a volume Avt, its energy is $= \frac{1}{2}rw^2a^2Avt$, hence its power density= $\frac{1}{2}rw^2a^2v$

To find decibels… $10 \times \log_{10}\left(\frac{power}{10^{-12}}\right)$

EM waves:

All the maths leads to the equation: $\frac{\partial^2 E}{\partial x^2} = e_0 m_0 \frac{\partial^2 E}{\partial t^2}$

From Maxwell's equations.

Note that their velocity= $c = \frac{1}{\sqrt{e_0 m_0}}$ (in vacuum!)

Spectrum: radio – IR – UV – X-rays – gamma-rays (long – short wavelength)

Polarisation: apply field E_0, you get a field $E = \frac{E_0}{1 + Na}$

Which all implies a change in the speed of light in a medium: $c = \frac{1}{\sqrt{e_0 e_r m_0 m_r}}$

e_r is the dielectric constant.

Refractive index: $n = c/c' = \sqrt{e_r m_r}$.

EMR travels slower in mediums due to effective weakening of E and B fields.

Group & phase velocity:

$$v_p = \frac{w}{k} \qquad v_g = \frac{dw}{dk}$$

A wave described by $\cos(wt - kx)$ or $\sin\left[\frac{2p}{l}(x - vt)\right]$, such that

$v_p = l/T = fl = w/k$, and refractive index $n = c/v = c/v_p$

The group velocity is the velocity of the envelope – the information speed.

If two waves: $w_1, k_1, f_1, l_1, w_2, k_2, f_2 l_2$, then, writing $w = \frac{w_1 + w_2}{2}$ and $k = \frac{k_1 + k_2}{2}$,

and $\Delta w = \frac{w_1 - w_2}{2}$ and similarly for Δk ; and if the waves are superimposed as:

$e^{i(k_1 x - w_1 t)} + e^{i(k_2 x - w_2 t)}$, then:

$$
\begin{aligned}
& e^{i(k_1 x - w_1 t)} + e^{i(k_2 x - w_2 t)} \\
&= e^{i(kx + \Delta kx - wt - \Delta wt)} + e^{i(kx - \Delta kx - wt + \Delta wt)} \\
&= e^{i(kx - wt)}\left[e^{i(\Delta kx - \Delta wt)} + e^{i(-\Delta kx + \Delta wt)}\right] \\
&= 2e^{i(kx - wt)}.\cos(\Delta k - \Delta wt)
\end{aligned}
$$

(Remember that $\cos q = \frac{e^{iq} + e^{-iq}}{2}$)

The exponential is an infinite wave, velocity $\frac{w}{k}$, and the cosine term has velocity $\frac{\Delta w}{\Delta k}$.

This implies that the cosine term is the envelope – small wavenumber (k) and frequency, so long wavelength.

If v_p is a constant, then $v_p = v_g$.

Otherwise, $\frac{dw}{dk} \neq \frac{w}{k}$, giving a refractive index n, which depends upon l .

So: $\frac{dn}{dl} = \frac{dn}{dk}.\frac{dk}{dl} = \frac{d}{dk}\left(\frac{c}{v_p}\right)\frac{d}{dl}\left(\frac{2p}{l}\right)$; remembering that $v_p = k/w$ and dw/dk is not a constant.

After algebra, get:

$$v_g = v_p\left(1 + \frac{l}{n}\frac{dn}{d\,l}\right)$$

Waves in 2/3D:

For 1D, string tension T, density m Kg/m, then:

$$\frac{\partial^2 f}{\partial x^2} = \frac{m}{T}\frac{\partial^2 f}{\partial t^2}\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad \left[v = \sqrt{\frac{T}{m}}\right]$$

In 2D, m Kg/m^2, then the wave equation becomes:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

To solve, use separation of variables; use $f(x, y, t) = f_x(x) f_y(y) f_z(z)$, and note that $\frac{\partial^2}{\partial x^2}$ only effects f_x. so equation becomes:

$$f_y\, f_t \frac{\partial^2 f_x}{\partial x^2} + f_x\, f_t \frac{\partial^2 f_y}{\partial y^2} = \frac{m}{T} f_x\, f_y \frac{\partial^2 f_t}{\partial t^2}$$

Then, divide by initial function: $f_x f_y f_t$:

$$\frac{1}{f_x}\frac{\partial^2 f_x}{\partial x^2} + \frac{1}{f_y}\frac{\partial^2 f_y}{\partial y^2} = \frac{m}{T}\frac{1}{f_t}\frac{\partial^2 f_t}{\partial t^2}$$

Note now, that the RHS is independent of x,y, and is then a constant: $-k$.
This is similar for the x and y parts of the LHS, call the constants $-k_x^2, -k_y^2$.
Then have 3 equations:

$\frac{\partial^2 f_a}{\partial t^2} = -k_a^2 f_a$ for $a = x, y$ and $\frac{1}{v^2}\frac{\partial^2 f_t}{\partial t^2} = -k^2 f_t$

With solutions $f_a = C_a e^{\pm i k_a a}$ and $f_t = C_t e^{\pm iwt}$ remembering that $k^2 = k_x^2 + k_y^2$.

So, $f = f_x f_y f_t = Ce^{i(k_x x + k_y y - wt)} = Ce^{i(\underline{k}\cdot\underline{r} - wt)}$

With $l_i = \frac{2p}{k_i}$

Generalise into 3D:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = \nabla^2 f$$

The Doppler effect:

Moving source:
Source moves with speed v. Wave travels at speed c, and it emitted at frequency f. And wavelength l .
Then, observed, with primes…
$l' = \left(1 - \tfrac{v}{c}\right) l$, and as $c = fl$ then

$$f' = \frac{1}{1 - \frac{v}{c}} . f = \frac{c}{c - v} f$$

Moving observer:

$$f' = \left(1 + \tfrac{v}{c}\right) f$$

Relativistic version: $f' = \dfrac{1}{1 - \frac{v}{c}} f . \dfrac{1}{g} = \sqrt{\dfrac{c + v}{c - v}} f$

Huygens construction:

Consider each point on a wavefront as a source of waves, which give another wavefront – but only forward wave needs to be considered – new wavefront is the common tangent.

Cases to be considered:

Plane Wave;
Circular Wave;
Reflection: $q_i = q_r$

Refraction: $\dfrac{\sin q_i}{\sin q_r} = \dfrac{v}{v'} = n$

Single slit: Plane wave strikes a screen, which absorbs everything except one point – waves become circular waves.

Double slit:

Circular wave patterns interfere.
Path difference $d \sin q$ Constructive interference at $d \sin q = nl$

Phase difference $d = \dfrac{2pd \sin q}{l}$

Amplitude received is proportional to $1 + e^{id} = e^{\frac{id}{2}} \underbrace{\left(e^{-\frac{id}{2}} + e^{\frac{id}{2}} \right)}_{2\cos d/2}$

So, intensity = |Amplitude|$^2 \propto 4\cos^2\left(\dfrac{d}{2}\right)$

Note, that $4 = 2^2$, with two slits!
Has peaks at $d/2 = np$.

Many slits: $1 + e^{id} + e^{2id} + ... + e^{i(N-1)d}$, which is a geometric series, with sum $\frac{e^{iNd} - 1}{e^{id} - 1} = \frac{\sin \frac{Nd}{2}}{\sin \frac{d}{2}}$

Giving peaks at $d = 2pn$

One wide slit: slit, width *a*, going from *–a/2* to *+a/2*. for any q and *x*, the path difference is always $\frac{2px \sin q}{l}$

So, total amplitude is proportional to $\int_{-a/2}^{+a/2} e^{\frac{2pix \sin q}{l}} dx$, which eventually gives a sinc function: *sinc(u) = sin(u)/u*.

Polarisation:

Vibrations in horizontal and vertical components (*H* and *V*).

Polaroid blocks out one component – so two, at 90deg will block out all.
In reflection of light, more *H* is reflected than *V*. Sunglasses work by blocking *H*, but allowing *V* to pass.
The colour of the atmosphere is due to scattering – and the light given off is polarised.
Birefringence:
Molecules are polarised (here, it means aligned…) by an *E* field, giving dielectric constant e_r, and a different speed of light. This effect also separates the *V* from *H* components, giving different e_r^H, e_r^V, n_V, n_H - so two images are seen.

Circular polarisation arises when the *V* and *H* components are out of step, say:
$E_H = E_0 \cos wt$ and $E_V = E_0 \sin wt$, then $E = \sqrt{E_H^2 + E_V^2} = E_0^2$.

Water Waves:

Wave a height *h* from bottom, to equilibrium position of wave, which is at , which is at *y = 0*. Wave velocity *c*. wave has components *v* and *u* in vertical and horizontal directions.

Assumptions:
1) nothing happens in z-direction;
2) water is incompressible;
3) ignore viscosity – no friction/damping;
4) amplitudes small: $A << l$ and $A << h$;
5) neglect surface tension;
6) vertical acc'n is small – "long waves in shallow water" – tidal waves.

Use pressure differences, areas and differences in height, to derive:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{gh} \frac{\partial^2 f}{\partial t^2} \qquad \text{so, } c^2 = gh$$

Solving differential equations:

How to find solutions of the equation:

$$\frac{d^2 f}{dt^2} + b\frac{df}{dt} + cf = 0$$

Remembering that *f(t)*, and *b*, *c* are constants

So, think of above equation as:

$$\left(\frac{d^2}{dt^2} + b\frac{d}{dt} + c\right)f = 0$$

$\Rightarrow$ with: $r_1 r_2 = c$ and $r_1 + r_2 = -b$

$$\left(\frac{d}{dt} - r_1\right)\left(\frac{d}{dt} - r_2\right)f = 0 \qquad \text{- (1)}$$

So, by the quadratic formula:

$$r_1 = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

Now, if you write: $g(t) = \left(\frac{d}{dt} - r_2\right)f(t)$

So, (1) becomes: $\left(\frac{d}{dt} - r_2\right)g(t) = 0 \Rightarrow \frac{d.g(t)}{dt} = r_1 g(t)$

Which you can solve by direct integration:

$\int \frac{dg}{g} = \int r_1 dt$ giving: $\ln g = r_1 t + const$

Therefore:

$$g = e^{r_1 t + const} = e^{r_1 t} e^{const} = Ae^{r_1 t}$$

So, now need to find *f*.

Now, $\left(\frac{d}{dt} - r_2\right)f = g = Ae^{r_1 t}$

So, $\frac{df}{dt} - r_2 f = Ae^{r_1 t}$

Which can be solved by the *integrating factor* method:

$$I = e^{-\int r_2 dt} = e^{-r_2 t}$$

Multiply by *I*:

$$e^{-r_2 t}\frac{df}{dt} - r_2 e^{-r_2 t} f = Ae^{(r_1 t - r_2 t)} \qquad \text{- (2)}$$

Remember that the product rule states:

$$\frac{d}{dt}(uv) = u\frac{dv}{dt} + v\frac{du}{dt}$$

So, (2) becomes:

$$\frac{d}{dt}\left(e^{-r_2 t} f\right) = Ae^{(r_1 - r_2)t}$$

Which can be integrated, giving:

$$e^{-r_2 t} f = \frac{A}{r_1 - r_2} e^{(r_1 - r_2)t} + B$$

Which you can express as:

$e^{-r_2 t} f = Ce^{(r_1 - r_2)t} + B$ if $r_1 \neq r_2$

So, therefore:

$$f(t) = Ce^{r_1 t} + De^{r_2 t}$$

Which is the solution!

Note, that for r_1, r_2 it doesn't matter which we choose for the positive or negative sign!

How to find solutions of the equation:

$$\frac{d^2 f}{dt^2} + b\frac{df}{dt} + cf = F(t) \qquad - (3)$$

Now, suppose f_{PI} solves (3) (particular integral)

And f_{CF} solves $\frac{d^2 f}{dt^2} + b\frac{df}{dt} + cf = 0$ (the complementary function)

So, $(f_{PI} + f_{CF})$ also solves (3); because:

$$\left(\frac{d^2}{dt^2} + b\frac{d}{dt} + c\right)(f_{PI} + f_{CF}) = F(t) + 0 = F(t)$$

So, any particular integral, plus the complementary function will give a general solution.

Suppose $F(t) = F_0 \cos(\Omega t)$ and $f \sim e^{i\Omega t}$

So, $f_{PI}(t) = Ae^{i\Omega t}$, with $A = \dfrac{F_0/m}{\Omega^2 - w^2 + igw}$ which contains no arbitrary constants!

So, the complete function: $f = f_{complete} = f_{PI} + f_{CF}$

$$f_{complete} = Ae^{i\Omega t} + \underbrace{Be^{-\frac{g}{2} + iw_0 t} + Ce^{-\frac{g}{2}t - iw_0 t}}_{CF}$$

Note, the $Ae^{i\Omega t}$ has no damping term – is not damped.

The B and C terms have gamma's – they are damped.

After time $t >> 1/g$ they vanish!

These are the *transients.*

At large t, can forget about the transients.

But cannot at small times.

$F(t) = F_0 \cos \Omega t \qquad t \geq 0$

$F(t) = 0 \qquad t < 0$

$f(t) = 0$ at $t = 0$ and before.

$f'(t) = 0$ at $t = 0$

so, can rewrite $f_{complete}$ as:

$$f_{complete} = Ae^{i\Omega t} + e^{-i\frac{g}{2}}\left(Be^{iw_0 t} + Ce^{-iw_0 t}\right)$$

Can find *B, C* by the initial conditions.

Matter waves:

Photoelectric effect: waves come in quanta of specific energies… $E = hf$ or $E = \hbar w$

From relativity: $E^2 = p^2c^2 + m_0^2c^4$ for photons $m_0 = 0$.

So $E = pc \Rightarrow p = E/c = hf/c = h/l = \hbar k$.

And, for Compton Scattering… photons energy E scatter off electrons.

E', p' are for the scattered photon, and E_e, p_e for the scattered electron.

Then, conservation of momentum says: $\underline{p} = \underline{p'} + \underline{p'_e}$

And conservation of energy: $E + m_ec^2 = E' + E_e$

After loads of algebra, get:

$$E' = \frac{m_ec^2E}{m_ec^2 + E(1-\cos q)}$$

This lot implies that we should treat photons as particle-like: the usage of $E = hf$.

So, how about particles behaving like waves…?

If particles are waves, they need a wavelength and frequency: $f = E/h$, $l = h/p$

Or, $E = \hbar w$ & $p = \hbar k$

So, we should be able to see diffraction and interference effects, if dimensions of 'slit' are comparable to the wavelength.

Which is of the order of $l \sim 10^{-34}m$ for a 1kg ball.

But, for an electron: $l \sim 10^{-3}m$.

There is a little confusion when talking about group/phase velocities, but if you stick to the particle velocity as being the group velocity, everything is fine.

Now, waves can be described by a wave equation: $y(x,t) = e^{i(kx-wt)}$.

Which describes a free particle wave – no boundaries/refraction…

So what about interacting particles…?

Suppose they are described by some $y(x,t)$… what do we know about p, E?

From the original wave equation: $y(x,t) = e^{i(kx-wt)}$, to get $\hbar w, \hbar k$:

$$\left(i\hbar\frac{\partial}{\partial t}\right)y = \hbar w y = Ey$$

$$\left(-i\hbar\frac{\partial}{\partial x}\right)y = \hbar k y = py$$

So, if we write: $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ and $\hat{E} = i\hbar\frac{\partial}{\partial t}$ which are just operators.

So, we have:

$$\hat{p}y = py \quad \text{and} \quad \hat{E}y = Ey$$

Usually: $E = PE + KE = V(x) + \frac{p^2}{2m}$ $\left\{\frac{1}{2}mv^2 = \frac{p^2}{2m}\right\}$

So: $Ey = V(x)y + \frac{p^2}{2m}y$

Therefore:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 y(x,t)}{\partial x^2} + V(x)y = i\hbar\frac{\partial y(x,t)}{\partial t}$$

Which is the Schrödinger wave-equation for matter.