# Maxwell's Equations:

Displacement current:

$$i_D = e_0 \frac{\partial f}{\partial t}$$

Thus, Amperes law becomes Maxwell's 3rd equation:

$$\nabla \times \underline{B} = m_0 \underline{j} + m_0 e_0 \frac{\partial \underline{E}}{\partial t}$$

Hence, all of Maxwell's equations:

$$\nabla \cdot \underline{E} = \frac{r}{e_0} \qquad \nabla \cdot \underline{B} = 0$$

$$\nabla \times \underline{B} = m_0 \underline{j} + m_0 e_0 \frac{\partial \underline{E}}{\partial t} \qquad \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

Or, in an alternative form:

$$\nabla \cdot \underline{D} = r_f$$

$$\nabla \times \underline{H} = \underline{j}_f + \frac{\partial \underline{D}}{\partial t}$$

# EM-Waves:

Maxwell's equations in free space read:

$$\nabla \cdot \underline{E} = 0 \qquad \nabla \cdot \underline{B} = 0$$

$$\nabla \times \underline{B} = m_0 e_0 \frac{\partial \underline{E}}{\partial t} \qquad \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

For the *z*-direction, say… they reduce to:

$$\frac{\partial E}{\partial z} = -\frac{\partial E}{\partial t} \qquad \frac{\partial B}{\partial z} = -m_0 e_0 \frac{\partial E}{\partial t}$$

Thus, get a 1D wave equation:

$$\frac{\partial^2 E}{\partial z^2} = m_0 e_0 \frac{\partial^2 E}{\partial t^2} \qquad \frac{\partial^2 B}{\partial z^2} = m_0 e_0 \frac{\partial^2 B}{\partial t^2}$$

Or, in 3D:

$$\nabla^2 \underline{E} = m_0 e_0 \frac{\partial^2 E}{\partial t^2}$$

And:

$$c^2 = \frac{1}{m_0 e_0}$$ which is the speed of EMR in vacuum

Solutions to wave equations have the form:

$$f(z - vt)$$

e.g:

$$\underline{E} = \underline{E}_0 e^{i(\underline{k}\cdot\underline{r} - wt)}$$

Relation between $\underline{E}$ and $\underline{B}$:

$$\underline{B} = \frac{1}{c}\hat{\underline{k}} \times \underline{E} \qquad \Rightarrow \qquad B_y = \tfrac{1}{c} E_x$$

Where $\hat{\underline{k}}$ is a unit vector in the direction of propagation.

Energy density in the wave is given by:

$$U = \frac{1}{2}\int_V (\underline{E}\cdot\underline{D} + \underline{B}\cdot\underline{H})dt$$

Or:

$$U = \frac{1}{2}\left(e_0 E^2 + \frac{1}{m_0}B^2\right)$$

$$= \frac{1}{2}\left(e_0 E_{0_x}^2 + \frac{1}{m_0}B_{0_y}^2\right)\cos^2(kz - wt)$$

$$E_{0_x} = cB_{0_y}$$

$$= \frac{1}{2}\left(e_0 E_{0_x}^2 + \frac{1}{m_0 c^2}E_{0_x}^2\right)\cos^2(kz - wt)$$

$$= e_0 E_{0_x}^2 \cos^2(kz - wt)$$

Thus, electric and magnetic fields carry the same amount of energy.

## Poynting Vector:

$\underline{N}$ $\qquad$ $Wm^{-2}$

Energy flux – energy crossing unit area per second.

$$N = cU\left(= cE_{0_x}^2 e_0 \cos^2(kz - wt)\right)$$

Thus:

$$N = \frac{1}{m_0} E_x B_y \left(= E_x H_y\right)$$

$$\boxed{\underline{N} = \frac{1}{m_0} \underline{E} \times \underline{B}} \qquad \left(= \underline{E} \times \underline{H}\right)$$

e.g.:

$$N = \frac{1}{m_0 c} E_0^2$$

Impedance of free space:

$$= \left(\sqrt{\frac{e_0}{m_0}}\right)^{-1} = 377\Omega$$

## Irradiance:

- a time averaged Poynting vector

$$< \underline{N} > = \frac{1}{2m_0} \frac{E_0^2}{c}$$

## Radiation Pressure:

| | |
|---|---|
| Absorbed: | $\underline{P} = \frac{< \underline{N} >}{c}$ |
| Reflected: | $\underline{P} = \frac{2 < \underline{N} >}{c}$ |

2 from conservation of momentum

Force exerted: $F = PA$ pressure times area

Reflection from a conductor: No $\underline{E}$-field inside… thus, reflected and incident wave must cancel at boundary… thus change of sign and direction of wave.

$E_T + E_R = 0$ at boundary.

## Polarisation:

### Plane/Linear Polarisation:

Electric field confined to a single plane

If the phase difference between components is $p$ , and the amplitudes are the same, then they are in phase.

## Circular/Elliptical Polarisation:

$$\underline{\hat{x}}E_{0_x}\cos(kz-wt)+\underline{\hat{y}}E_{0_y}\cos(kz-wt+d)$$

Any phase difference other than $p$ will produce elliptical polarisation, with a phase difference of $p/2$ giving circular polarisation:

Right Circular:

$$\underline{\hat{x}}E_{0_x}\cos(kz-wt)+\underline{\hat{y}}E_{0_y}\sin(kz-wt)$$

Left Circular:

$$\underline{\hat{x}}E_{0_x}\cos(kz-wt)-\underline{\hat{y}}E_{0_y}\sin(kz-wt)$$

A phase shifter – or retarder – will add phase differences.

# EM Waves in Materials:

$$\nabla\cdot\underline{D}=r_f \qquad \nabla\cdot\underline{B}=0$$

$$\nabla\times\underline{E}=-\frac{\partial\underline{B}}{\partial t} \qquad \nabla\times\underline{H}=\underline{j}+\frac{\partial\underline{D}}{\partial t}$$

## Dielectrics:

$$e_0\rightarrow ee_0$$

$$m_0\rightarrow mm_0$$

Assume no charges/currents. Thus, the wave equations become:

$$\nabla^2\underline{E}=mm_0ee_0\frac{\partial^2\underline{E}}{\partial t^2}$$

Similarly for $\underline{B}$.

Now:

$$v=\frac{1}{\sqrt{mm_0ee_0}}=\frac{c}{\sqrt{me}}$$

But: $v=\frac{c}{n}$

Thus:

$$n=\sqrt{me}\sim\sqrt{e}$$

As $m\sim 1$ for most dielectrics.

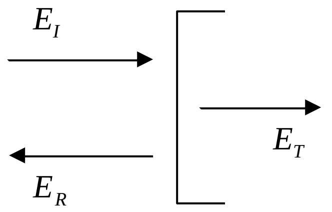
Snell's Law:

$$n_1 \sin q_1 = n_2 \sin q_2$$

$$k_I \sin q_I = k_R \sin q_R = k_T \sin q_T$$

- generally

From boundary conditions.




Going from vacuum into a dielectric

Boundary conditions say:

$$E_I + E_R = E_T$$
$$H_I + H_R = H_T$$

Thus:

$$\frac{B_I}{m_0} - \frac{B_R}{m_0} = \frac{B_T}{mm_0}$$

And: $B = \frac{1}{c}E = \sqrt{e_0 m_0}\,E$

Hence:

$$E_I\sqrt{\frac{e_0}{m_0}} - E_R\sqrt{\frac{e_0}{m_0}} = E_T\sqrt{\frac{ee_0}{mm_0}}$$

Using the approximation above, and putting in the refractive index:

$$E_I - E_R = nE_T$$

Hence, the reflection and transmission coefficients are:

$$R = \frac{E_R^2}{E_I^2} = \left(\frac{1-n}{1+n}\right)^2 \qquad T = \frac{E_T^2}{E_I^2}\frac{v}{c} = \frac{4n}{(1+n)^2}$$

Note that: $T + R = 1$

## Conducting Medium:

$$\underline{j} = s\,\underline{E}$$

"Ohm's Law"

Hence, Maxwell 3 becomes:

$$\nabla \times \underline{B} = m_0 s\, \underline{E} + e_0 m_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

In a good conductor:

$$m_0\, \underline{j} >> e_0 m_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

Thus, the equation to work with is:

$$\nabla \times \underline{B} = m_0 s\, \underline{E}$$

## Skin Depth:

$$d = \sqrt{\frac{2}{m_0 s w}}$$ an attenuation factor.

Thus, the resistance becomes:

$$R = R_0 \frac{r}{2d}$$

## Plasmas:

Using: $\underline{F} = m\underline{\ddot{r}}$ $\underline{F} = q\underline{E}$ $\underline{j} = Ne\underline{\dot{r}}$
Can get:

$$\nabla^2 \underline{E} = m_0 n_e e \left( \frac{e\underline{E}}{m_e} \right) + e_0 m_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

Below the "plasma frequency":

$$w_p = \sqrt{\frac{n_e e^2}{e_0 m_e}}$$

Incident electromagnetic waves are strongly attenuated.
Derived from finding the complex $k$.

*Phase Velocity:*

$$v_p = \frac{w}{k}$$

*Group Velocity:*

$$v_g = \frac{dw}{dk}$$