

Lagrangian:

$$\underline{v} = \dot{r}\hat{r} + r\dot{\mathbf{q}}\hat{\mathbf{q}} \quad \text{in polars}$$

Small amplitude oscillations: $\mathbf{q} \ll 1$

$$\sin \mathbf{q} \sim \mathbf{q} \quad \cos \mathbf{q} \sim 1 - \frac{\mathbf{q}^2}{2}$$

Solutions to the SHM equations:

$$\mathbf{q}_i = A_i e^{i\omega t}$$

Torque: $\underline{\mathbf{t}} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$

Angular momentum: $\underline{\mathbf{L}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}}$

$$\frac{d\underline{\mathbf{L}}}{dt} = \underline{\mathbf{t}}$$

Kinetic energy: $T = T_{trans} + T_{rot}$

$$T_{trans} = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{r}^2$$

$$T_{rot} = \frac{1}{2}I\Omega^2 \quad v = r\Omega \Rightarrow \Omega = \frac{v}{r}$$

$$\text{Thus: } T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I\frac{\dot{r}^2}{r^2} = \frac{1}{2}\dot{r}^2\left(m + \frac{I}{r^2}\right)$$

The rotation creates a “mass correction” term.

D'Alembert's principle:

System of 'n'-particles, with 'S'-constraints, hence, the number of degrees of freedom 'N' is:

$$N = 3n - S$$

Get N generalised coordinates.

Lagrange Equation:

$$L(q_i, \dot{q}_i, t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \quad L = T - U$$

Conservation of angular momentum:

$$\frac{d}{dt}(mr^2\dot{\mathbf{q}}) = 0$$

Properties:

Homogeneity of time:

Lagrangian of a closed system does not depend upon time:

$$E = T + V = \text{const}$$

Homogeneity of time \rightarrow Conservation of energy.

Homogeneity of space:

Lagrangian of a closed system remains unchanged under parallel displacement of the whole system in space.

$$\underline{p} = \sum_i m_i \underline{v}_i = \text{const}$$

Homogeneity of space \rightarrow Conservation of linear momentum.

Isotropy of space:

Lagrangian of a closed system is invariant under rotation transformations.

$$\sum_i \underline{r}_i \times \underline{p}_i = \text{const}$$

Isotropy of space \rightarrow Conservation of angular momentum.

If symmetry shows that rotation about z -axis invariant, then L_z is conserved.

Noether's theorem: Symmetry of system \rightarrow Respective conservation law.

Calculus of Variations:

A path length between two points is given by, in Cartesian space:

$$S = \int_A^B ds$$

$$ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2} = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx \sqrt{1 + (y')^2}$$

Hence:

$$S = \int_A^B \sqrt{1 + (y')^2} dx \quad y' = \frac{dy}{dx}$$

Thus: $S = \int_A^B f(x, y, y') dx$

$$I\{y(x)\} = \int_{x_1}^{x_2} f(x, y, y') dx \quad y' = \frac{dy}{dx}$$

And, Euler's equation can be used to find the extremum:

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

Now, for the Cartesian $f(x, y, y') = \sqrt{1 + (y')^2}$:

$$\frac{\partial f}{\partial y} = 0 \quad \frac{\partial f}{\partial y'} = \frac{1}{2} (1 + (y')^2)^{-\frac{1}{2}} 2y' = \frac{y'}{\sqrt{1 + (y')^2}}$$

Hence:

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}} = \text{const} = a$$

Hence: $y' = \frac{dy}{dx} = \text{const} = a \quad \Rightarrow \quad y = ax + b$

A straight line.

Principle of Least Action

$$A = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt \quad - \text{ like Euler's equation, but with the Lagrangian.}$$

Evaluate the integral over a whole period: $2\pi/w$.

To minimise the action, with respect to the amplitude of a vibration c_i :

$$\frac{\partial A}{\partial c_i} = 0$$

Hamiltonian:

$$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - L$$

Generalised momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

If $\frac{\partial H}{\partial t} = 0$, then $H = E \dots$ conservation of energy