Fourier Series:

Sines & Cosines:

Example:

$$f(x) = b_1 \sin\frac{px}{L} + b_2 \sin\frac{2px}{L} + \ldots$$

Or:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{npx}{L}$$

Now, a property of sines is that:

$$\int_{-L}^{+L} \sin\frac{npx}{L} \sin\frac{mpx}{L}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

On the interval $-L \le x \le +L$

Now, to find the coefficient b_1, multiply both sides by $\sin\frac{px}{L}$, and integrate:

$$\int_{-L}^{+L} \sin\frac{px}{L} f(x)dx = \int_{-L}^{+L} b_1 \sin\frac{px}{L} \sin\frac{px}{L} dx + \int_{-L}^{+L} b_2 \sin\frac{2px}{L} \sin\frac{px}{L} dx + \ldots$$

Which is all zero, except the first term, which is half the interval length. Hence:

$$b_1 = \frac{1}{L} \int_{-L}^{+L} \sin\frac{px}{L} f(x)dx$$

Hence, one should see, more generally:

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin\frac{npx}{L} dx \qquad n = 1,2,3,\ldots$$

Similarly for cosines:

If:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{npx}{L}$$

Then:

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)dx$$

$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos\frac{npx}{L} dx \qquad n = 1,2,3,\ldots$$

Technically, there is a b_0 term – but it is zero. Note that the a_0 isn't zero, and is divided by *2L*.
Now, for a general function:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{npx}{L} + b_n \sin\frac{npx}{L}\right) \qquad \text{on} - L \le x \le +L$$

Where:

$$b_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\frac{npx}{L}dx$$

$$a_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)dx \qquad n = 1,2,3,\ldots$$

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\frac{npx}{L}dx$$

Note that you divide a_n & b_n by half the length of the interval/integral.

For Fourier series to work, *f(x)* must satisfy ***Dirichlet's conditions***:

- *f(x)* must be single valued & have finite number of discontinuities;
- $\int_{-L}^{+L} |f(x)|dx$ must be finite

Hence the Fourier series will *converge* to *f(x).*
Fourier series use the property of the orthogonality between sines and cosines:

$$\int_{-L}^{+L} \cos\frac{npx}{L}\sin\frac{mpx}{L}dx = 0$$

Complex Exponentials:

Two complex functions are orthogonal on $a \le x \le b$ if:

$$\int_a^b u^*(x)v(x)dx = 0$$

Hence, the two complex exponentials are orthogonal:

$$\int_{-L}^{+L}\left(e^{\frac{inpx}{L}}\right)^*\left(e^{\frac{impx}{L}}\right)dx = \begin{cases} 0 & m \ne n \\ 2L & m = n \end{cases}$$

Thus, if:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{inpx}{L}}$$

Then: $\qquad$ for $-L \le x \le +L$

$$c_n = \frac{1}{2L}\int_{-L}^{+L} e^{-\frac{inpx}{L}dx}$$