

Separation of Variables

The equation:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$$

Has solutions of the form:

$$f(x,t) = X(x)T(t)$$

So, evaluating the derivatives:

$$\frac{\partial^2 f}{\partial x^2} = T \frac{d^2 X}{dx^2} \quad \frac{\partial^2 f}{\partial t^2} = X \frac{d^2 T}{dt^2}$$

Hence, putting back into PDE:

$$T \frac{d^2 X}{dx^2} = \frac{1}{c^2} X \frac{d^2 T}{dt^2}$$

Dividing by $f = XT$:

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2} \frac{1}{T} \frac{d^2 T}{dt^2}$$

These can only be equal if:

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -k^2 = \frac{1}{c^2} \frac{1}{T} \frac{d^2 T}{dt^2}$$

Hence giving two normal differential equations. Which can be solved easily.

Now, if $X_1(x)T_1(t)$ & $X_2(x)T_2(t)$ are solutions, then by superposition, so is:

$$f = X_1(x)T_1(t) + X_2(x)T_2(t)$$

In fact, so is:

$$f(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$$

For a particular set of boundary conditions, a solution is:

$$f(x,t) = \sum_{n=1}^{\infty} \sin \frac{np\pi x}{L} \left(A_n \cos \frac{np\pi ct}{L} + B_n \sin \frac{np\pi ct}{L} \right)$$

And the coefficients A_n & B_n can be found via Fourier series.

The solution $X_n(x)$ is an eigenfunction, with eigenvalue k_n^2 . This is because it conforms to the eigenproblem:

$$\frac{d^2}{dx^2} X_n = -k_n^2 X_n$$

When coupled with boundary conditions.

A useful relationship, to tidy expressions up, is:

$$\cos(n\pi) = (-1)^n \quad n = 1, 2, 3, \dots$$

Now, two eigenfunctions & eigenvalues for a problem satisfy:

$$\frac{d^2}{dx^2} X_n = -k_n^2 X_n$$

$$\frac{d^2}{dx^2} X_m = -k_m^2 X_m$$

Now, it can be shown that:

$$\int_0^L X_n X_m dx = 0 \quad m \neq n$$

Which implies that the eigenfunctions X_n & X_m are orthogonal. Hence, eigenfunctions are orthogonal.

Laplace's Equation:

$$\nabla^2 f = 0$$

Under the boundary conditions:

- $f(x, 0) = f(x, L) = f(0, y) = 0$
- $f(L, y) = V$

Which is a square box, with all but one side at potential 0, and the other at V .

The solutions are of the form:

$$f(x, y) = \sum_{n=1}^{\infty} \left(A_n e^{\frac{n\pi x}{L}} + B_n e^{-\frac{n\pi x}{L}} \right) \sin \frac{n\pi y}{L}$$

And use Fourier series (for complex exponentials) to find the coefficients.

The Heat Flow Equation:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{D} \frac{\partial f}{\partial t} \quad \text{where } D = \frac{k}{Cr} \quad \begin{array}{l} \text{- thermal diffusivity} \\ \mathbf{k} = \text{thermal conductivity} \end{array}$$

And $f(x, t)$ gives the temperature at (x, t) .

The boundary conditions:

$$\left. \frac{\partial f}{\partial x} \right|_0 = \left. \frac{\partial f}{\partial x} \right|_L = 0$$

Over a length L , give a solution of the form:

$$f(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-g_n t}$$

Where: $g_n = D \left(\frac{n\pi}{L} \right)^2$ "relaxation rate"

Hence can find coefficients from cosine Fourier series.