Derive the material derivative

Now, if:

$$r(x, y, z, t) \tag{1.1}$$

Then:

$$dr = \frac{\partial r}{\partial t}dt + \frac{\partial r}{\partial x_i}dx_i \tag{1.2}$$

Divide (1.2) by dt, and take limit:

$$\lim_{dt\to 0}\left(\frac{dr}{dt}\right) = \frac{\partial r}{\partial t} + \lim_{dt\to 0}\left(\frac{dx_i}{dt}\right) \tag{1.3}$$

Now:

$$\lim_{dt\to 0}\left(\frac{dx_i}{dt}\right) = \frac{\partial x_i}{\partial t} = u_i \tag{1.4}$$

We define the LHS of (1.3) as the material derivative:

$$\lim_{dt\to 0}\left(\frac{dr}{dt}\right) \equiv \frac{Dr}{Dt} \tag{1.5}$$

Hence, putting all together:

$$\frac{Dr}{Dt} = \frac{\partial r}{\partial t} + u_i\frac{\partial r}{\partial x_i} \tag{1.6}$$

Or, expanding out of suffix notation:

$$\boxed{\frac{Dr}{Dt} = \frac{\partial r}{\partial t} + (\underline{u}\cdot\nabla)r} \tag{1.7}$$

- the material derivative

Derive the integral form of the continuity equation

Now, the mass in volume *V*:

$$\int_V r\,dV \tag{2.1}$$

Rate of change of mass in *V*:

$$\frac{d}{dt}\int_V r\,dV = \int_V \frac{\partial r}{\partial t}dV \qquad (2.2)$$

The rate of mass flow out of *V*, through a bounding closed surface *S*:

$$\int_S r\,\underline{u}\cdot\underline{n}dS \qquad (2.3)$$

Hence, if mass is conserved, (2.2) & (2.3) must balance:

$$\int_V \frac{\partial r}{\partial t}dV = -\int_S r\,\underline{u}\cdot\underline{n}dS \qquad (2.4)$$

- the integral form of mass the continuity equation

Using divergence theorem & mass conservation in integral form, derive the pointwise form of mass conservation:

Divergence theorem:

$$\int_V (\nabla\cdot\underline{a})dV = \int_S \underline{a}\cdot\underline{n}dS \qquad (3.1)$$

Integral form of mass conservation:

$$\frac{d}{dt}\int_V r\,dV = -\int_S r\,\underline{u}\cdot\underline{n}dS \qquad (3.2)$$

Now, the RHS of (3.2), using (3.1) becomes:

$$\int_S r\,\underline{u}\cdot\underline{n}dS = \int_V (\nabla\cdot r\,\underline{u})dV \qquad (3.3)$$

Now, expanding out the divergence in the RHS of (3.3), by vector calculus:

$$\nabla\cdot(r\,\underline{u}) = r(\nabla\cdot\underline{u}) + (\underline{u}\cdot\nabla)r \qquad (3.4)$$

Hence, (3.3) becomes:

$$\int_S r\,\underline{u}\cdot\underline{n}dS = \int_V \{(\underline{u}\cdot\nabla)r + r(\nabla\cdot\underline{u})\}dV \qquad (3.5)$$

Now, putting (3.5) into (3.2), and bringing over to the LHS:

$$\int_V \frac{\partial r}{\partial t}dV + \int_V \{(\underline{u}\cdot\nabla)r + r(\nabla\cdot\underline{u})\}dV = 0 \tag{3.6}$$

Which can be bought under a single integral. The volume can be shrunk to a point. Hence:

$$\frac{\partial r}{\partial t} + (\underline{u}\cdot\nabla)r + r(\nabla\cdot\underline{u}) = 0 \tag{3.7}$$

Which contains elements of the material derivative:

$$\frac{\partial r}{\partial t} + (\underline{u}\cdot\nabla)r \equiv \frac{Dr}{Dt} \tag{3.8}$$

Hence,

$$\boxed{\frac{Dr}{Dt} + r(\underline{u}\cdot\nabla)r = 0} \tag{3.9}$$

- the pointwise form of the continuity equation

Derive equation for hydrostatic equilibrium

Now, suppose a body force $\underline{F}(\underline{x},t)$ per unit mass, then the total body force is:

$$\int_V r\,\underline{F}dV \tag{4.1}$$

Now, the total internal pressure force is:

$$-\int_S p\underline{n}dS \tag{4.2}$$

If the system is in equilibrium, (4.1) & (4.2) must balance. Hence:

$$\int_V r\,\underline{F}dV = \int_S p\underline{n}dS \tag{4.3}$$

Now, if we use the divergence theorem in the form:

$$\int_V \nabla f dV = \int_S f\,\underline{n}dS \tag{4.4}$$

Thus, the RHS of (4.3) becomes:

$$\int_S p\underline{n}dS = \int_V \nabla p dV \tag{4.5}$$

Hence, (4.3) becomes:

$$\int_V (r\,\underline{F} - \nabla p)\,dV = 0 \qquad (4.6)$$

We can shrink the volume down to a point, hence, taking over to the other side:

$$\boxed{r\,\underline{F} = \nabla p} \qquad (4.7)$$

- the equation for hydrostatic equilibrium

Derive Euler's equation

Now, if we start from Newton's 2nd law:

$$\underline{F} = \frac{d\,\underline{p}}{dt} = \frac{d}{dt}(m\underline{v}) = m\frac{d\,\underline{v}}{dt} + \underline{v}\frac{dm}{dt} \qquad (5.1)$$

The analogue is:

$$\frac{d}{dt}\left\{\int_V \underline{u}\,r\,dV\right\} = -\int_S r\,\underline{u}(\underline{u}\cdot\underline{n})dS + \int_V r\,\underline{F} - \nabla p\,dV \qquad (5.2)$$

Or, rate of change of momentum is mass flux in plus the resultant force. Now, if looking at the middle integral in (5.2):

$$\int_S r\,\underline{u}(\underline{u}\cdot\underline{n})dS = \left(\int_S ru(\underline{u}\cdot\underline{n})dS, \int_S rv(\underline{u}\cdot\underline{n})dS, \ldots\right) \qquad (5.3)$$

Using the divergence theorem on one of the components of (5.3):

$$\int_S ru(\underline{u}\cdot\underline{n})dS = \int_V \nabla\cdot(ru\underline{u})dV \qquad (5.4)$$

And, expanding out the RHS of (5.4):

$$\int_V \nabla\cdot(ru\underline{u})dV = \int_V \{u\nabla\cdot(r\,\underline{u}) + r\,(\underline{u}\cdot\nabla)u\}dV \qquad (5.5)$$

Thus, doing this for all components of the RHS of (5.3), the LHS becomes:

$$\int_S r\,\underline{u}(\underline{u}\cdot\underline{n})dS = \int_V \{\underline{u}(\nabla\cdot r\,\underline{u}) + r\,(\underline{u}\cdot\nabla)\underline{u}\}dV \qquad (5.6)$$

Hence, the original equation, (5.2) becomes:

$$\int_V \frac{\partial}{\partial t}\left(r\,\underline{u}\right)dV + \int_V \left\{\underline{u}\left(\nabla \cdot r\,\underline{u}\right) + r\left(\underline{u} \cdot \nabla\right)\underline{u}\right\}dV - \int_V \left\{r\,\underline{F} + \nabla p\right\}dV \quad (5.7)$$

Which can all be put under a single integral:

$$\int_V \left\{ r\,\frac{\partial \underline{u}}{\partial t} + \underline{u}\,\frac{\partial r}{\partial t} + \underline{u}\left(\nabla \cdot r\,\underline{u}\right) + r\left(\underline{u} \cdot \nabla\right)\underline{u} - r\,\underline{F} + \nabla p \right\} dV = 0 \quad (5.8)$$

Now, notice that some of the components can be simplified:

$$r\,\frac{\partial \underline{u}}{\partial t} + r\left(\underline{u} \cdot \nabla\right)\underline{u} \equiv r\,\frac{D\underline{u}}{Dt} \quad (5.9)$$

$$\underline{u}\left(\frac{\partial r}{\partial t} + \nabla \cdot r\,\underline{u}\right) = 0 \quad (5.10)$$

(5.10) by the continuity equation. Hence:

$$\int_V \left\{ r\,\frac{D\underline{u}}{Dt} - r\,\underline{F} + \nabla p \right\} dV = 0 \quad (5.11)$$

We can shrink the volume down to a point, and thus (5.11) becomes:

$$r\,\frac{D\underline{u}}{Dt} - r\,\underline{F} + \nabla p = 0 \quad (5.12)$$

Or:

$$\boxed{\frac{D\underline{u}}{Dt} = \underline{F} - \frac{1}{r}\nabla p} \quad (5.13)$$

- Euler's equation

From Euler's equation, derive Bernoulli's equation

$$\frac{D\underline{u}}{Dt} = -\frac{1}{r}\nabla p + \underline{F} \quad (6.1)$$

Now:

$$\frac{1}{r}\nabla p = \frac{1}{r}\frac{\partial p}{\partial x_i} = \frac{d}{dp}\left[\int \frac{dp}{r}\right]\frac{\partial p}{\partial x_i}$$

We can do this as r is "barotropic"

$$= \frac{\partial}{\partial x_i} \int \frac{dp}{r}$$

$$= \nabla \left(\int \frac{dp}{r} \right)$$

Hence:

$$\frac{1}{r} \nabla p = \nabla \left(\int \frac{dp}{r} \right) \qquad (6.2)$$

Now, as $\underline{F}$ is a "conservative force", we can associate a scalar potential:

$$\underline{F} = -\nabla \Omega \qquad (6.3)$$

Now, writing out the definition of the material derivative:

$$\frac{D\underline{u}}{Dt} \equiv \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} \qquad (6.4)$$

Now, $\underline{u}$ is a "steady flow". Thus:

$$\frac{\partial \underline{u}}{\partial t} = 0 \qquad (6.5)$$

Now, there is a vector identity:

$$(\underline{u} \cdot \nabla)\underline{u} = \nabla\left(\tfrac{1}{2} \underline{u} \cdot \underline{u}\right) - \underline{u} \times \nabla \times \underline{u} \qquad (6.6)$$

We shall define the vorticity vector, $\underline{w} \equiv \nabla \times \underline{u}$. Hence, putting (6.6) on the LHS of (6.1); and using (6.2) & (6.3) on the RHS… (5.1) becomes:

$$\nabla\left(\tfrac{1}{2} \underline{u} \cdot \underline{u}\right) - \underline{u} \times \underline{w} = -\nabla \left(\int \frac{dp}{r} \right) - \nabla \Omega \qquad (6.7)$$

Which can be bought under a common grad:

$$\nabla \left(\tfrac{1}{2} \underline{u} \cdot \underline{u} + \int \frac{dp}{r} + \Omega \right) = \underline{u} \times \underline{w} \qquad (6.8)$$

Now, if the flow is irrottational, i.e. $\underline{w} = \underline{0}$, then:

$$\boxed{\frac{1}{2} \underline{u} \cdot \underline{u} + \int \frac{dp}{r} + \Omega = const} \qquad (6.9)$$

- Bernoulli's equation

Prove:

$$\nabla(f\underline{a}) = f(\nabla\cdot\underline{a}) + (\underline{a}\cdot\nabla)f \quad \text{(V1.1)}$$

Now, in suffix notation, using the chain rule for differentiation:

$$\frac{\partial}{\partial x_i}(fa_i) = f\frac{\partial a_i}{\partial x_i} + a_i\frac{\partial f}{\partial x_i} \quad \text{(V1.2)}$$

And, putting back into vector notation:

$$\nabla\cdot(f\underline{a}) = f(\nabla\cdot\underline{a}) + (\underline{a}\cdot\nabla)f \quad \text{(V1.3)}$$

Use the divergence theorem to prove that:

$$\int_V \nabla f dV = \int_S f\underline{n}dS \quad \text{(V2.1)}$$

Divergence theorem:

$$\int_V (\nabla\cdot\underline{a})dV = \int_S \underline{a}\cdot\underline{n}dS \quad \text{(V2.2)}$$

Now, let $\underline{a} = f\underline{c}$, where $\underline{c}$ an arbitrary constant vector. Thus:

$$\nabla\cdot\underline{a} = \nabla\cdot(f\underline{c}) = f(\nabla\cdot\cdot\underline{c}) + (\underline{c}\cdot\nabla)f \quad \text{(V2.3)}$$

Now, due to the definition of $\underline{c}$, $\nabla\cdot\underline{c} = 0$. Thus:

$$\nabla\cdot\underline{a} = (\underline{c}\cdot\nabla)f \quad \text{(V2.4)}$$

Thus, (V2.2) becomes:

$$\int_V (\underline{c}\cdot\nabla)f dV = \int_S f\underline{c}\cdot\underline{n}dS \quad \text{(V2.5)}$$

Hence:

$$\underline{c}\cdot\int_V \nabla f dV = \underline{c}\cdot\int_S f\underline{n}dS \quad \text{(V2.6)}$$

Thus, we have proven:

$$\int_V \nabla f dV = \int_S f\underline{n}dS \quad \text{(V2.7)}$$

Prove:

$$\nabla\times(f\underline{a}) = f(\nabla\times\underline{a}) + (\nabla f)\times\underline{a} \tag{V3.1}$$

Now, look at LHS:

$$\nabla\times f\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ fa_1 & fa_2 & fa_3 \end{vmatrix} \tag{V3.2}$$

Now, looking only at the '*i*' component:

$$\left.\nabla\times f\underline{a}\right|_i = \frac{\partial}{\partial y}(fa_3) - \frac{\partial}{\partial z}(fa_2) \tag{V3.3}$$

Which can be expanded by the chain rule:

$$\begin{aligned} \left.\nabla\times f\underline{a}\right|_i &= f\frac{\partial a_3}{\partial y} + a_3\frac{\partial f}{\partial y} - f\frac{\partial a_2}{\partial z} - a_2\frac{\partial f}{\partial z} \\ &= f\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) + \left(a_3\frac{\partial f}{\partial y} - a_2\frac{\partial f}{\partial z}\right) \\ &= f\left.(\nabla\times\underline{a})\right|_i + \left.(\nabla f)\times\underline{a}\right|_i \end{aligned}$$

Now, all other components will be similar. Hence:

$$\nabla\times(f\underline{a}) = f(\nabla\times\underline{a}) + (\nabla f)\times\underline{a} \tag{V3.4}$$

Prove:

$$\underline{u}\times\nabla\times\underline{u} = \nabla\left(\tfrac{1}{2}\underline{u}\cdot\underline{u}\right) - (\underline{u}\cdot\nabla)\underline{u} \tag{V4.1}$$

Now, looking at part of the LHS, with $\underline{u} = (u, v, w)$:

$$\begin{aligned} \nabla\times\underline{u} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} \\ &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\underline{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\underline{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\underline{k} \end{aligned}$$

$$\left.\underline{u}\times\nabla\times\underline{u}\right|_i = \left.\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u & v & w \\ \left.\nabla\times\underline{u}\right|_i & \left.\nabla\times\underline{u}\right|_j & \left.\nabla\times\underline{u}\right|_k \end{vmatrix}\right|_i$$

$$= v\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + w\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) \qquad \text{(V4.2)}$$

For convenience, define:

$$\begin{array}{lll} u_x = \dfrac{\partial u}{\partial x} & u_y = \dfrac{\partial u}{\partial y} & u_z = \dfrac{\partial u}{\partial z} \\ v_x = \dfrac{\partial v}{\partial x} & v_y = \dfrac{\partial v}{\partial y} & \ldots \\ w_x = \dfrac{\partial w}{\partial x} & \ldots & \ldots \end{array}$$

Hence, (V4.2) becomes:

$$\begin{aligned} \underline{u} \times \nabla \times \underline{u}\big|_i &= vv_x - vu_y + ww_x - wu_z \\ &= (uu_x + vv_x + ww_x) - (uu_x + vu_y + wu_z) \\ &= \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{1}{2}w^2\right) - (\underline{u} \cdot \nabla)\underline{u} \end{aligned}$$

Thus, putting together, and other components working similarly:

$$\underline{u} \times \nabla \times \underline{u} = \nabla\left(\tfrac{1}{2}\underline{u} \cdot \underline{u}\right) - (\underline{u} \cdot \nabla)\underline{u} \qquad \text{(V4.3)}$$