The Physics of Neutron Stars

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Abstract

The aim of this project was to use the 4^{th} order Runge-Kutta numerical method to determine various properties of a neutron star. Two equations of state were used: a parameterisation from Bethe & Johnson and one derived using the density of states for a Fermi-gas.

The model used neutron stars in equilibrium, thus the equation for hydrostatic equilibrium was solved.

There are two models used for the equilibrium equations: a classical approximation, and the general relativity treatment of the Tolman-Oppenheimer-Volkov equation.

The maximum mass of a neutron star was calculated to be $1.8M_{\odot}$ and has a radius of 9.7km.

The maximum radius of a neutron star was calculated to be 11.7km, with a mass of $1.07M_{\odot}$.

These results were found using the Tolman-Oppenheimer-Volkov equation, with Bethe & Johnson's equation of state.

1. Introduction

Since the discovery of the neutron, it has been hypothesised that bodies with very high densities may exist in a neutron degenerate state. Since then, objects have been found that might fall under this category. The properties of such objects are of much academic debate, due to the lack of accurate experimental data.

Due to the fact that it is hard to obtain such data, it is very important to investigate these objects via theoretical, computational and mathematical methods; modelling the interactions of exotic states of matter under such extreme conditions.

During this report we shall investigate different models that predict the nature of neutron stars. A number of mathematical methods were derived to obtain more accurate numerical results.

Values for constants and units used can be found in the appendix.

2. Origin of Neutron Stars

First predicted in 1934 by W.Baade & F.Zwickey[1], neutron stars remained purely an academic curiosity until 1967, when J.Bell & A.Hewish[2] discovered the neutron star in the form of a *radio pulsar* – a rapidly rotating neutron star. This discovery was actually accidental, with the sources of the radio pulses subject to much speculation, including the (rumoured) name of the source *LGM1* – Little Green Men – until similar sources had been found.

In a stable star, thermonuclear radiation pressure and gravity will balance, leaving the star with a regular size. However, when radiation pressure falls – i.e. at the end of a stars lifetime – then gravity will cause the star to collapse.

The outcome of the collapse will depend on the mass of the core of the star.

Stars with core masses below the Chandrasekhar limit of 1.44 $M_\odot$ become electron-degenerate white dwarfs[3].

Stars with cores between the Chandrasekhar limit and the Tolman-Oppenheimer-Volkov limit[4] of ~1.4 - 3$M_\odot$ will become neutron stars. It has been proposed that a "quark star" is the result of a core mass above the TOV limit[5], although these have not been observed yet. A black hole is the product of cores with mass above the TOV limit. There is an uncertainty of the TOV limit, due to the uncertain nature of the equations of state of matter at such high densities.

Initial mass	Last stage in stellar evolution	Final product	Final mass
< 6 $M_\odot$	Planetary nebula	White dwarf	< 1.4 $M_\odot$
6 - 9 $M_\odot$	Supernova	White dwarf/neutron star	< 1.4 $M_\odot$
9 - 40 $M_\odot$	Supernova	Neutron star	< 2 - 3 $M_\odot$
> 40 $M_\odot$	Supernova	Black hole	> 2 - 3 $M_\odot$

Table 1. This data has been adapted from M.Demianski[6] to illustrate the final stages of stellar evolution, with the mass ranges involved.

Considering only neutron star formation, we continue:

It is more energetically viable for the constituent protons and electrons to combine via inverse beta-decay, forming more neutrons[7].

The core will thus become a solid neutron-degenerate ball. When the outer gaseous layers of the star fall into the centre, they impact the central solid "neutron star" ball and then rebound. However, the very outer layers of the star are still falling inwards. These two shells of stellar material collide, releasing immense amounts of energy which manifests itself as a supernova. These supernova can have a luminosity greater than that of its host galaxy. The ejected material continues to move away from the central neutron star, forming the nebulous patterns we can observe.

Due to the anisotropic nature of stellar collapse, when the stellar material hits the surface of the neutron star, the central ball is given a "kick", due to conservation of angular momentum; and is usually manifested as a high-frequency rotation. The now rapidly rotating neutron star becomes known as a pulsar, with typical periods of rotation of the order $\sim 10^{-3}\,\mathrm{s}$. These pulsars are now able to be detected on the earth. Another result of the anisotropic collapse is that the central neutron star will be given lateral motion – which can send the neutron star flying off into space at relative speeds of up to 1500km/s.

Examples of such neutron stars are SN1054 – better known as the Crab Nebula, which has a central pulsar; and PuppisA[8] – which has a central neutron star moving at a relative speed of 1500km/s.

When the supernova exploded which produced the Crab Nebula, in 1054, it was the second brightest object in the night sky – the first being the Moon. The Crab Nebula is 6300 light years away, yet the Chinese were able to view the supernova with the naked eye for around 2 years after the explosion. Considering only ~10% of a supernovas energy is manifested as the emission of optical energy[9] one can see how energetic they are.

The composition of the resulting neutron star is badly understood, although it has been proposed that hyperons – fermions with specific properties - exist at its centre, with a surrounding sea of superfluid fermions, a crust of solid neutrons; and a possible 1m thick atmosphere of Hydrogen or Helium. This is purely theoretical speculation[10].

3. Hydrostatic Equilibrium

We shall consider a neutron star in equilibrium. That is to say, the inward gravitational force is balanced by an outward pressure. This pressure originates from the degenerate nature of neutrons – which are a member of the class of particles known as fermions. We shall assume that the neutron stars do not rotate; and that its mass is symmetrically and spherically distributed throughout the system.

If the temperature of a "ball" of fermions is taken towards the limit of zero, the energy of the system will go below a limit known as the Fermi energy. Below this energy, the neutrons become degenerate. Thus, they fill up all possible energy levels, placing the system into its ground state. As the Pauli Exclusion Principle[11] states that no two fermions can occupy the same quantum state, there will be a finite size associated with such a close packing of degenerate fermions. If such a structure of degenerate neutrons is compressed (e.g. by gravity), then, as the neutrons cannot be any closer together (by Pauli) a "neutron degeneracy pressure" is thus described.

Hence, as the inward gravitational force works to shrink the degenerate fermion "ball", the outward neutron degeneracy pressure keeps the system at a particular size.

Using a Newtonian gravitational model, the following equation for the force due to gravity can be written:

$$F = -\frac{Gm(r)}{r^2} r(r) \qquad (3.1)$$

Where: G is the universal gravitational constant; $m(r)$ is the total mass inside a radius r, and $r(r)$ the mass-density at radius r.

$m(r)$ can be found from the volume integral under spherical coordinates, which has the following form† due to spherical symmetry:

$$m(r) = 4p \int_0^r r(r) r^2 dr \qquad (3.2)$$

Hence, a differential equation may be constructed:

$$\frac{dm}{dr} = 4pr^2 r(r) \qquad (3.3a)$$

Now, if the system is in equilibrium, and an outward pressure force P is present, the following equation must hold‡:

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2} r(r) \qquad (3.3b)$$

Thus, equations (3.3a) and (3.3b) form a coupled set of first order differential equations.

There is an analogue to (3.3b), which uses Einstein's theory of gravitation, known as the Tolman-Oppenheimer-Volkov (TOV) equation[12]:

$$\frac{dP}{dr} = -\left(r + \frac{P}{c^2}\right)\frac{G\left(m + \frac{4p}{c^2} Pr^3\right)}{r^2\left(1 - \frac{2Gm}{c^2 r}\right)} \qquad (3.4)$$

The solutions to these equations will be explored.

4. Rescaling Equations

To solve system (3.3), it is useful to first create dimensionless quantities, to reduce numerical rounding errors. We shall scale relevant quantities thus:

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†The general equation for the mass inside a volume V is given by: $m(r) = \int_V r(r) dV$

‡ This is a form of the easily derivable general equation of hydrostatic equilibrium: $\nabla P = r \underline{F}$

$$\hat{r} = \frac{r}{R_0} \tag{4.1a}$$

$$\hat{m} = \frac{m}{M_0} \tag{4.1b}$$

$$\hat{P} = \frac{P}{r_s} \tag{4.1c}$$

$$\hat{r} = \frac{r}{r_s} \tag{4.1d}$$

Where:

r_s = density at $r = 0$: the central density

$$R_0 \equiv \frac{1}{\sqrt{4pGr_s}} \tag{4.2a}$$

$$M_0 \equiv \frac{4pr_s}{\left(\sqrt{4pGr_s}\right)^3} \tag{4.2b}$$

Hence, system (3.3) becomes:

$$\begin{cases} \dfrac{d\hat{m}}{d\hat{r}} = \hat{r}^2 \hat{r} & (4.3a) \\ \dfrac{d\hat{P}}{d\hat{r}} = -\dfrac{\hat{m}\hat{r}}{\hat{r}^2} & (4.3b) \end{cases}$$

The TOV equation (3.4), under the scaled units of (4.1) gives:

$$\frac{d\hat{P}}{d\hat{r}} = -\frac{\left(\hat{P} + \hat{r}\right)\left(\hat{r}^3 \hat{P} + \hat{m}\right)}{\hat{r}^2 - 2\hat{m}\hat{r}} \tag{4.4}$$

The system of equilibrium equations (4.3a) and (4.3b) will be referred to as the *classical model*; (4.3a) and (4.4) the *relativistic model*.

5. Equation of State

As well as solving the equations for equilibrium, there is also a requirement for how the density r is dependant upon the pressure P. These dependences are known as the *equation of state*. The first one to be employed is a parameterisation from Bethe & Johnson[13]:

$$P(n) = 363.44 \times n^{2.54} \tag{5.1a}$$

$$r(n) = nm_n + 236 \times n^{2.54} \tag{5.1b}$$

Where n is the number density, m_n the mass of the neutron. In the formulation of this equation of state, it was assumed the temperature of the system is at absolute zero; and was only designed to work in the density range $95 \le r \le 6170\text{MeV.fm}^{-3}$. This restriction[14] has been placed on all data. From now on, (5.1) shall be referred to as the *BJ equation of state*.

Another equation of state may be derived[§], using the density of states[15] for fermions:

$$P(n) = 79.37 \times n^{\frac{5}{3}} \qquad (5.2a)$$

$$r(n) = nm_n + 119 \times n^{\frac{5}{3}} \qquad (5.2b)$$

In deriving the equation of state (5.2) it was assumed that the fermions were all non-interacting, non-relativistic neutrons, and that the temperature of the system was in the limit of absolute zero.

6. Computational Method

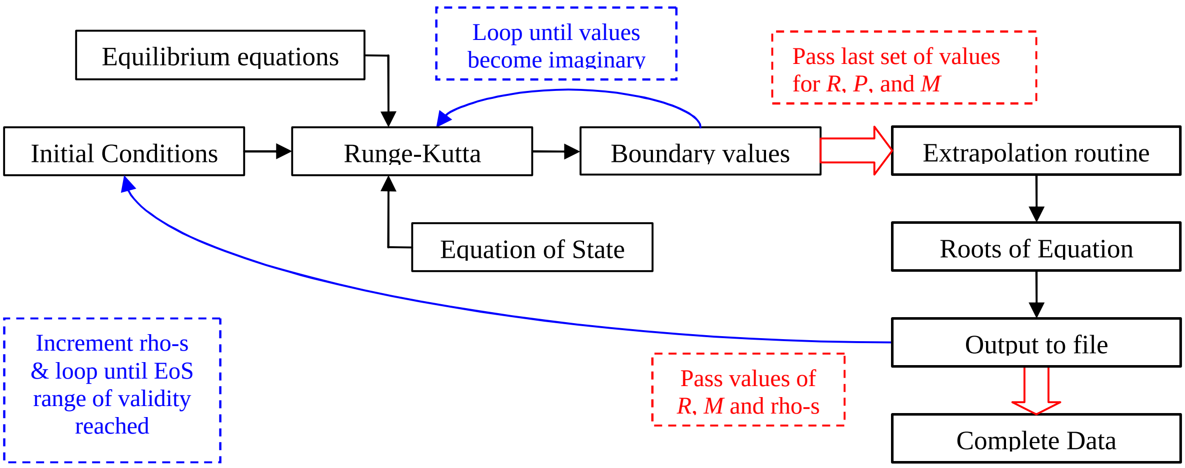

Fig 1. Schematic illustrating computational method

The *initial conditions* are a set of parameters for an infinitesimal volume at the centre a neutron star; an arbitrary central density r_s, with both mass and radius being zero. These conditions are used to initiate the 4th order *Runge-Kutta* (RK4) numerical algorithm for solving the *equilibrium equations* by incrementing the radius by a small amount, the step size *h*, whilst utilising the *equation of state*.

This process is looped until the *boundary* of the star is reached, i.e. when $P = 0$. This however does not occur at a well defined point, and due to the numerical calculations in the equation of state, values will become complex after this point. Thus the exact radius and mass cannot be found at the boundary $P = 0$.

To obtain an estimation of the radius and mass at the boundary, *extrapolation routines* were used to fit an n^{th}-order polynomial to the last $(n+1)$ computed points. Linear, quadratic and cubic fits were derived for this purpose. The roots of the pressure-radius polynomial are found, and the

[§] The form of the equations derived are: $P(n) = \dfrac{n^{\frac{5}{3}} 3^{\frac{2}{3}} p^{\frac{4}{3}} (\hbar c)^2}{5 m_n}$ and $r(n) = nm_n + \dfrac{(\hbar c)^2 (3p^2 n)^{\frac{5}{3}}}{10 p^2 m_n}$

correct root chosen. This radius was fed into a mass-radius extrapolation, and the corresponding mass computed.

This whole process is repeated with a new central density, until the density reaches its maximum in the range of validity (see **§**5).

6.1 4th Order Runge-Kutta

These techniques[16] were developed around 1900 by the German mathematicians after whom the process is named.

The system of equations

$$\frac{d\underline{x}}{dr} = f(\underline{x}, r) \tag{6.1}$$

can be approximately solved by employing a numerical method, with a set of initial conditions.

The RK4 method is dependant upon an increment – the step size *h* – on the independent parameter *r*. In essence, the method calculates the tangent to the actual function at regular intervals, four times per step. Weighted averages of these tangents are then calculated, thus predicting the next point on the solution curve. The method is initiated by starting with a set of initial conditions.

The step size will dictate how accurate the numerical solutions are; with global errors being of the order h^4.

7. Results

A programme was written in C++ to compute the method outlined in **§**6.

When the classical equilibrium equations were used in conjunction with the BJ equation of state, the following maxima were found:

R_{max} *km*	$M(R_{max})$ $M_\odot$	r_s $MeV.fm^{-3}$	M_{max} $M_\odot$	$R(M_{max})$ *km*	r_s $MeV.fm^{-3}$
16.7	12.9	2294	24.7	13.7	23967

Table 2. Maximum radius and mass, with corresponding parameters. This data has been found using a step size of 0.001563; with the classical equilibrium equations and the BJ equation of state.

The predicted maximum mass is above the TOV limit of ~$3M_\odot$. The central density is also well above the range of validity for the BJ equation of state. For this reason that we shall discard the classical model, and only use the relativistic one.

Fig 2 shows the results obtained from using the BJ equation of state, with the relativistic model.

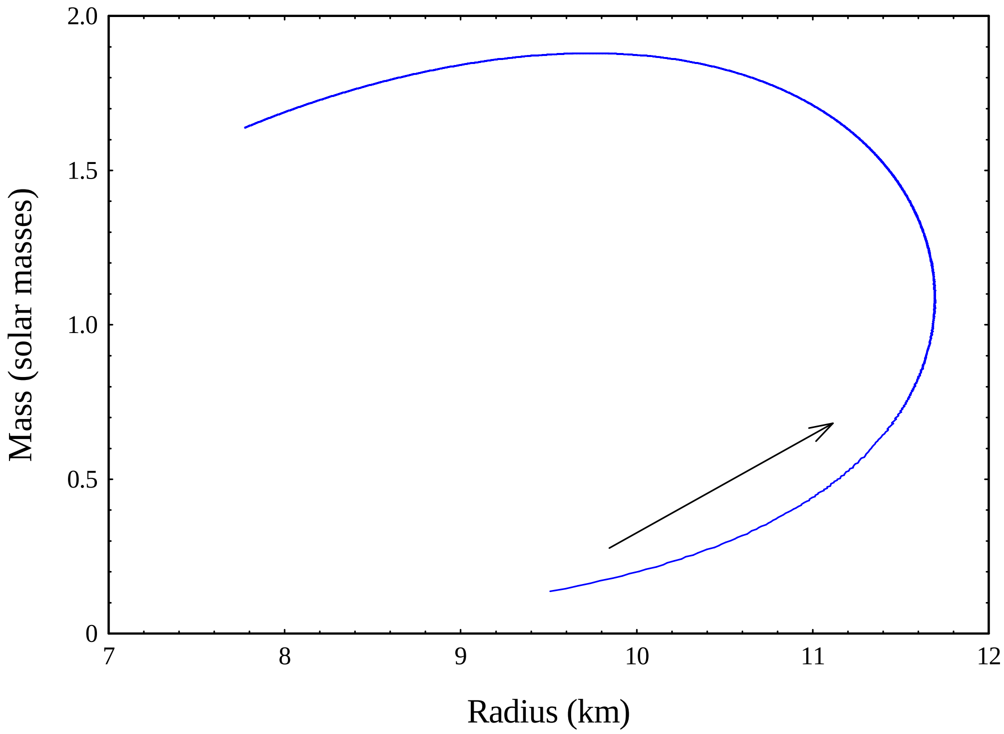

Fig 2. Mass-Radius curve: showing results obtained from using the relativistic model, under the linear extrapolation routines and using the BJ equation of state. The arrow indicates direction of increasing central density. With step size h = 0.0015625.

Fig 3. shows the mass-radius curve obtained from using the derived non-interacting non-relativistic equation of state; using the relativistic pressure model.

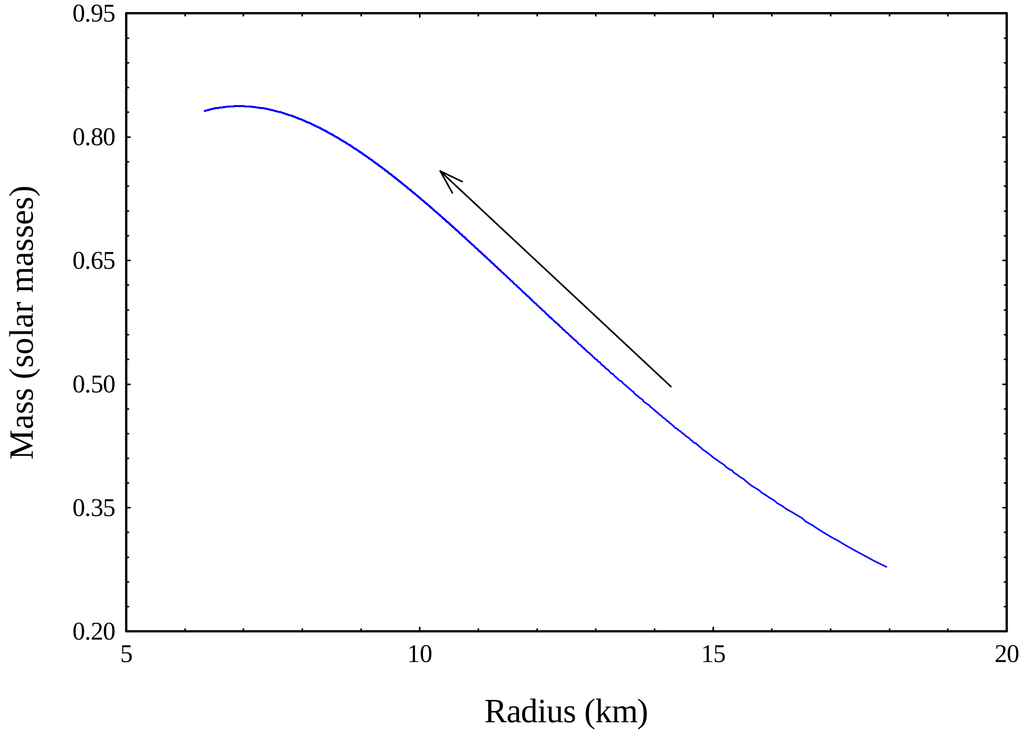

Fig 3. Mass-Radius curve: results obtained from using the relativistic model, with the derived equation of state; and under the linear extrapolation routine. The arrow indicates direction of increasing central density.
This has been computed with a step size h = 0.0015625

8. Results Evaluation

It is useful to note that in the limit of small central densities, the classical and relativistic pressure models, with the BJ equation of state, coincide under the RK4 numerical method. This is what is expected in an analytic evaluation of such a limit.

In the derivation of the equilibrium equations, it was assumed that the neutron stars do not rotate. If rotational effects are included[17], one finds that the maximum neutron star mass can increase by up to ~25%.

As a comparison to densities inside neutron stars, an estimate of nuclear density gives $129.7 MeVfm^{-3}$. This uses the approximate nuclear liquid-drop model.

The solutions are dependant upon two sources of error: that due to the RK4 inaccuracies, and that of the extrapolation routine.

The error due to RK4 is thus:

$$\pm h^4 R_0 \tag{8.1}$$

Where R_0 is defined by equation (4.2a).

The error due to an n^{th}-order polynomial extrapolation is,

$$\pm \sum_{i=n+1}^{\infty} x^i \approx \pm x^{n+1} \tag{8.2}$$

for $x < 1$, where x is the distance between two consecutive computed points immediately before the expected root. Notice that the error due to extrapolation decreases as the order of the fitted polynomial increases. But, at the same time the number of calculable points decreases with step size, as the fitted polynomial produces unusable results, due to the nature of the roots of the polynomial. Thus, at very small step sizes, it is better to use a linear fit, as there is little difference in accuracy between the linear and cubic extrapolations. Notice that both Fig 2 and Fig 3 use the linear extrapolation. At larger step size, it is better to use a higher-order polynomial fit.

Hence, the total error s_i on each parameter *i*, under an n^{th}-order polynomial extrapolation is thus:

$$s_i = \pm\left(h^4 R_0 + x^{n+1}\right)$$

(8.3a)
(8.3b)

Employing these errors, one finds the maximum mass, under the BJ equation of state, and the relativistic model. With corresponding radii. These use the linear extrapolation, as this provides the maximum number of data points, thus a much more conclusive maxima can be found, at small step size.

h	$M_{\max}$ - $M_\odot$	$R(M_{\max})$ - km	r_s - MeV.fm^{-3}
0.05	1.87812 ± 0.03985	9.7038 ± 0.3012	1665.37
0.025	1.87911 ± 0.00961	9.73414 ± 0.14934	1693.67
0.003125	1.87926 ± 0.00274	9.74184 ± 0.01855	1715.28

Table 3. Data showing the computed maximum mass of a neutron-degenerate object in equilibrium, as predicted by the BJ equation of state, under the relativistic model. With corresponding radii and central densities.

An equivalent set of values is found for the maximum radius, with corresponding masses; again under the BJ equation of state and the relativistic model.

h	R_{max} - km	$M(R_{max})$ - $M_\odot$	r_s - MeV.fm^{-3}
0.05	11.6015 ± 0.5747	1.17337 ± 0.15291	458.008
0.025	11.6467 ± 0.2973	1.10652 ± 0.02283	427.424
0.003125	11.695 ± 0.038	1.0717 ± 0.0005	412.496

Table 4. Data showing the computed maximum radii of a neutron-degenerate object in equilibrium, as predicted by the BJ equation of state, under the relativistic model. With corresponding masses and central densities

9. Conclusion

We can draw the following conclusion from the derived equation of state and the data computed from its employment: no neutron degenerate object, in equilibrium, exists with a mass above $0.83M_\odot$.

However, observed[18] neutron stars have masses within the region 1.2-$1.6M_\odot$, after taking into account the observational errors on such masses. Thus, on the initial basis of experimental observation, one may assume that there are inaccuracies in the derived equation of state. These inaccuracies could arise from the fact that it does not take into account inter-neutron interactions; or the possibility of different classes of fermions, or other particles existing within the star.

The BJ equation of state suggests that no neutron-degenerate object exists with a mass above $1.9M_\odot$, and is in equilibrium; and that no such object may have a radius above 11.7km. This conclusion is of course only valid for objects which satisfy the assumptions outlined in this report. This mass-limit is also within the previously mentioned Tolman-Oppenheimer-Volkov limit, and is consistent with observation.

10. Directions for Future Work

An interesting route for further work would be to derive dynamical models, investigating small perturbations of a neutron star around its equilibrium position. We propose that this would predict that the equilibrium equations will be a result when the minimum of the appropriate potential function is found. This will also be of interest if other minima of the potential function are found, and what they physically correspond to.

The derived equation of state used a binomial expansion of a square root, to give the non-relativistic equation of state. If a binomial expansion is not used, a much more accurate relativistic equation of state would be found; although this will be much harder to use, as a complicated function will be required to be inverted. Although this will still be of limited use, as it will not take into account inter-neutron interactions.

Appendix

A1. Constants & Units

Table A 1 shows the natural units used throughout, as well as the value of various constants used. Notice that r is actually an "energy density" due to its units, but is referred to as "density" throughout the report.

Quantity	Symbol	Unit/value
Mega-electron volt	*MeV*	$1\times10^{6}\times1.602\times10^{19}$ J
Fermi	*fm*	1×10^{15} m
Pressure	P	MeVfm^{-3}
Energy-density	r	MeVfm^{-3}
Number density	n	fm^{-3}
Mass	m	MeVc^{-2}
--	$\hbar c$	197.327MeVfm
Universal gravitational constant	G	$\hbar c\times6.67259\times10^{-45}$ MeVfm
Mass of the Sun	$M_\odot$	1.989×10^{30} kg
Mass of the neutron	m_n	938.926MeVc^{-2}

A 1. Table of units and constants used throughout. Values taken from M.Hjorth-Jensen[12].

Throughout the report, the "sol" symbol $M_\odot$ has been used to denote the mass of the sun.

A2. Fitting Polynomials

To fit an n^{th}-order polynomial:

$$y(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 \tag{A2.1}$$

One must have *(n + 1)* data points:

$$(x_1, y_1),(x_2, y_2),\ldots,(x_{n+1}, y_{n+1}) \tag{A2.2}$$

Hence, we may write the following system of equations:

$$\begin{aligned} a_n x_1^n + a_{n-1}x_1^{n-1} + \ldots + a_0 &= y_1 \\ a_n x_2^n + a_{n-1}x_2^{n-1} + \ldots + a_0 &= y_2 \\ &\cdots \\ a_n x_n^n + a_{n-1}x_n^{n-1} + \ldots + a_0 &= y_n \\ a_n x_{n+1}^n + a_{n-1}x_{n+1}^{n-1} + \ldots + a_0 &= y_{n+1} \end{aligned} \tag{A2.3}$$

Which can be compressed into an *(n+1)* by *(n+1)* matrix:

$$\begin{pmatrix} x_1^n & x_1^{n-1} & \dots & 1 \\ x_2^n & x_2^{n-1} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ x_{n+1}^n & x_{n+1}^{n-1} & \dots & 1 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \\ \dots \\ a_0 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{n+1} \end{pmatrix} \qquad \text{(A2.4)}$$

Now, to find the coefficients, one can invert the matrix. Thus:

$$\begin{pmatrix} a_n \\ a_{n-1} \\ \dots \\ a_0 \end{pmatrix} = \begin{pmatrix} x_1^n & x_1^{n-1} & \dots & 1 \\ x_2^n & x_2^{n-1} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ x_{n+1}^n & x_{n+1}^{n-1} & \dots & 1 \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{n+1} \end{pmatrix} \qquad \text{(A2.5)}$$

Hence, once the matrix has been inverted & the product found, one can write a polynomial which will exactly fit the *(n + 1)* points; provided the matrix can be inverted.

A3. Derivation of Equation of State

When a neutron star – neutrons are fermions – is constructed from degenerate fermions, they form systems with all fermions in the lowest energy levels. This "packing" of fermions has a particular form, which follows from the Pauli Exclusion Principle – no two fermions may be in the same quantum state.

A gas of fermions which is "cold enough" so that all states below the Fermi-energy are occupied, is called a *degenerate Fermi gas*. The Fermi energy is defined by e , and will be derived below.

Hence, a certain number of neutrons, under a certain density, at its "Fermi pressure" – the pressure at which the fermions become degenerate – will occupy a minimum volume in space.

Here, due to a unit-consideration, the momentum p and wavenumber k are used interchangeably; by the relation $p = \hbar k$, and setting $\hbar = 1$ for convenience.

In what follows, this space is known as "*k*" space.

The following derivation uses the assumption that the fermions are *non-interacting* & *non-relativistic*. It is done under the limit that the temperature tends to zero.

Consider a single quadrant of a shell of thickness dk of a sphere. In the region $k \to k + dk$, the quadrant-shell has a volume:

$$\frac{1}{8} 4pk^2 dk \qquad \text{(A3.1)}$$

Now, the volume of one state, in *k*-space is:

$$\left(\frac{p}{L}\right)^3 \tag{A3.2}$$

Hence, the density:

$$dN(k) = \frac{\frac{1}{8}4pk^2dk}{\left(\frac{p}{L}\right)^3} = \frac{L^3}{2p^2}k^2dk \tag{A3.3}$$

Now, define: $dn(k) \equiv \frac{dN(k)}{V}$ to be the density of states. Where $V = L^3$

Hence:

$$dn(k) = \frac{1}{2p^2}k^2dk \tag{A3.4}$$

Thus,

$$n = 2\int_0^{k_F} \frac{1}{2p^2}k^2dk \tag{A3.5}$$

Giving:

$$n = \frac{k_F^3}{3p^2} \tag{A3.6}$$

Gives the total number of states, up to the total of k_F, at the Fermi-energy. The multiplicative factor of 2 is due to the fact that neutrons are fermions; whence each “state” can actually have two spin states.

Now, the energy is given by:

$$E = \sqrt{p^2c^2 + m^2c^4} \tag{A3.7}$$

Under our model, the momentum $p = \hbar k$, and for units-sake, $c = \hbar = 1$. Hence, the expression for the energy becomes, after expansion to second order:

$$e(k) = \sqrt{k^2 + m^2} \approx m + \frac{k^2}{2m} \tag{A3.8}$$

Now, the energy density r is the number of states multiplied by its energy. Thus, the total energy density of a system, up to its Fermi energy is:

$$\tilde{r}(k) = 2\int_0^{k_F} n(k)e(k)dk = 2\int_0^{k_F} \frac{k^2}{2mp^2}\left(m + \frac{k^2}{2m}\right)dk \qquad \text{(A3.9)}$$

Again, the factor of two due to each state being host to two spin states. This gives an expression for the energy density. $\tilde{r}$ denotes the function in a form we shall be modifying, thus a different functional dependence will be used. Hence:

$$\tilde{r}(k) = \frac{1}{p^2}\left[\frac{mk_F^3}{3} + \frac{k_F^5}{10m}\right] \qquad \text{(A3.10)}$$

However, substituting the form of k_F in terms of n from (A3.6):

$$r(n) = \frac{1}{p^2}\left[p^2 mn + \frac{\left(3p^2 n\right)^{\frac{5}{3}}}{10m}\right] \qquad \text{(A3.11)}$$

By thermodynamics, we have the relation:

$$P(n) = r\frac{\partial r}{\partial n} - r \qquad \text{(A3.12)}$$

Which gives:

$$P(n) = \frac{n^{\frac{5}{3}} 3^{\frac{2}{3}} p^{\frac{4}{3}}}{5m} \qquad \text{(A3.13)}$$

Hence, the derived equations of state are:

$$P(n) = \frac{n^{\frac{5}{3}} 3^{\frac{2}{3}} p^{\frac{4}{3}}}{5m} \qquad \text{(A3.14a)}$$

$$r(n) = \frac{1}{p^2}\left[p^2 mn + \frac{\left(3p^2 n\right)^{\frac{5}{3}}}{10m}\right] \qquad \text{(A3.15b)}$$

A more accurate relativistic non-interacting equation of state may be derived by re-evaluating the Fermi energy, without expanding the square root. This derivation has been slightly inadequate in keeping up with various factors of $\hbar c$, but once a dimensional analysis is done, one may write the following dimensionally consistent equations:

$$P(n) = \frac{n^{\frac{5}{3}} 3^{\frac{2}{3}} p^{\frac{4}{3}} (\hbar c)^2}{5m} \qquad \text{(A3.15a)}$$

$$r(n) = \frac{1}{p^2}\left[p^2 mn + \frac{\left(3p^2 n\right)^{\frac{5}{3}} (\hbar c)^2}{10m}\right] \qquad \text{(A3.16b)}$$

A4. References

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