Electrodynamics

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Abstract

These are a set of notes I have made, based on lectures given by R.Jones at the University of Manchester Jan-June '08. Please e-mail me with any comments/corrections: jap@watering.co.uk.

Contents

1 Linear Algebra & Introduction

This section just lays some of the groundwork & very brief mathematical framework for some of the things we will be using.

1.1 Basis Vectors

Suppose we define a set of vectors, such that:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

That is, they are of length one (normal), and are orthogonal (at 'right angles') to each other. Hence, we have a set of orthonormal basis vectors. The cartesian basis vectors are:

$$\boldsymbol{e}_1 = (1, 0, 0) \quad \boldsymbol{e}_2 = (0, 1, 0) \quad \boldsymbol{e}_3 = (0, 0, 1)$$

Using this basis, we are able to write vectors in terms of the basis. We use the equivalent notation:

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = (x_1, x_2, x_3) = x_i\boldsymbol{e}_i$$

Where the last line has used the Einstein summation convention of implied summations for repeated indices. We will come back to this later, when discussing tensors in *relativistic electrodynamics*.

1.2 Rotations & Matrices

In the subsequent discussion, I will be being sloppy with notation, and write the vector $\boldsymbol{x}$ as x.

Let us write down the operation of some matrix M on a vector x, giving us some new vector x':

$$\begin{aligned}
\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \\
M_{ij}x_j &= x'_i
\end{aligned}$$

The final line $M_{ij}x_j = x'_i$ is the index notation for matrix multiplication. The indices refer to individual components, irrespective of the basis used.
Now, the transpose of the matrix M is:

$$M^T_{ij} = M_{ji}$$

Notice then, that if:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Then:

$$x^T x = x^2 = (x_1, x_2, x_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 \end{pmatrix}$$

If a set of basis vectors $\boldsymbol{e}_i$ are rotated to some new orientation $\boldsymbol{e}'_i$, then the components of the vectors in the primed system (new ones) are related to those in the unprimed system (old ones) by $x' = Rx$, or:

$$x'_i = R_{ij}x_j$$

Where we have that $R_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j$, the angle between bases in the primed and unprimed frame. The cosines between the two frames. For a rotation about the z-axis, this looks like:

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The inverse of R obviously has elements $\boldsymbol{e}_i \cdot \boldsymbol{e}'_j$; which is just the transpose of R. Thus, $R^{-1} = R^T$. This sort of matrix is said to be orthogonal. Hence, orthogonal transformations preserve lengths and orientations of vectors within. This is equivalent to $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}' \cdot \boldsymbol{y}'$. This will become very important in our discussion on *relativistic electrodynamics*; in the invariance of quantities.

It is important that the distinction between the rotation matrix, R, and the operator M is understood.
Suppose we had that x is operated upon by some operator M, to give y:

$$y = Mx$$

Now, suppose we want to find the equivalent in the primed frame. So, let us rotate y, via some rotation matrix:

$$y' = Ry$$

Putting our expression for y in:

$$y' = R(Mx)$$

Now, a rotation followed by its inverse does nothing; its like saying, rotate by 90° and back again. Thus, $R^T R = 1$ (essentially). Hence, if we put this factor in, nothing is changed:

$$y' = R(MR^T Rx)$$

Let us collect the terms:

$$y' = RMR^T(Rx)$$

Now, the effect of x by rotation operator R is $Rx = x'$. Hence:

$$y' = RMR^T x'$$

Now, we also know that in the primed frame, transformations can still happen:

$$y' = M'x'$$

So, upon comparison:

$$M' = RMR^T$$

Hence, we have an expression for the operator M in the primed frame. In index notation this is:

$$M'_{ij} = R_{ik}M_{kl}R^T_{lj} = R_{ik}R_{jl}M_{kl}$$

Where in the first equality we wrote down the operation, in a way conducive to matrix multiplication; then used the transposed matrix element, and rearranged. The rearranged expression does not represent matrix multiplication. Infact, it is a prototype for how a cartesian tensor, of second rank, transforms.

1.2.1 Tensors

This is an impressively brief introduction to these objects!

A cartesian tensor is a some T_{ijkl} which transforms via:

$$T'_{abcd} = R_{ai}R_{bj}R_{ck}R_{dl}T_{ijkl}$$

This is for a rank-4 tensor. Notice where the indices are in the rotation matrices R_{ij}. The tensor T' can be thought of as the 'image' of T, after the set of transformations given by the rotation matrices R.
As another example, consider a rank-1 tensor: a vector. It will transform via:

$$T'_a = R_{ai}T_i$$

A matrix, which is a rank-2 tensor will then transform via:

$$T'_{ab} = R_{ai}R_{bj}T_{ij}$$

Which is what we saw in the previous section: how an operator M transformed.
It is easy to see how this generalises to a rank-n tensor.

We shall have a more complete discussion on tensors when we start to use them & their transformation properties. In the present discussion, we have not made a distinction between tensors in different spaces (as we shall do later).
The salient points to take away from this is that a tensor (which is a set of quantities T) transforms under a given set of rules (where the transformation has been denoted R here). That is, for a general rank-n tensor, it must transform like:

$$T'_{a_1 a_2 \dots a_n} = R_{a_1 b_1}R_{a_2 b_2}\dots R_{a_n b_n}T_{b_1 b_2 \dots b_n}$$

This will become clearer when we use them.

1.3 Brief Note on Important Vector Equations

The 'curl' of a vector is the measure of how much it 'rotates'. That is, something that has linear field lines, which just stretch off to infinity has no curl.
The 'div' of a vector is a measure of the divergence of a field from a source. The divergence of the magnetic field is zero - no magnetic monopoles.

The div grad $= \nabla \cdot \nabla\phi = \nabla^2\phi$.
The divergence of a curl of a vector field is zero. This can be visualised, as well as mathematically proved, as the curl gives you a field which is 'going round in circles', which will have no 'radial' component; hence its divergence is zero:

$$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$$

Stokes theorem:

$$\int_S \nabla \times \boldsymbol{V}\, dS = \oint_\ell \boldsymbol{V} \cdot d\boldsymbol{\ell}$$

That is, the measure of how many field lines pass through a closed path, enclosing a surface.

Divergence theorem:

$$\int_V \nabla \cdot \boldsymbol{V} dV = \int_S \boldsymbol{V} \cdot d\boldsymbol{S}$$

That is, the measure of flux of field lines, through a surface enclosing some volume within which the field exists.

We will be making extensive use of the following:

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{|r|}$$

We shall try to be consistent with the notation that $|r|$ is the magnitude of the vector $\boldsymbol{r}$, and that $\hat{\boldsymbol{r}}$ is a unit vector in the direction of $\boldsymbol{r}$.
It is easy to derive various relations, but one that we use a lot is:

$$\nabla \frac{1}{r} = -\frac{\hat{\boldsymbol{r}}}{|r|^2} = -\frac{\boldsymbol{r}}{|r|^3}$$

Where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, $|r| = r = \sqrt{x^2 + y^2 + z^2}$.

It is very useful to note that dot-products commute. That is:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

And also that cross-products anti-commute:

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

1.3.1 Important Electrodynamics Equations

Most of these will be derived, or used at some point; but this is just a recap.

Maxwell's equations:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{1.1}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{1.2}$$

In matter, these take on the following form:

$$\nabla \cdot \boldsymbol{D} = \rho_f \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{1.3}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t} \tag{1.4}$$

Where we have used the following definitions:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \tag{1.5}$$

$$\boldsymbol{H} = \frac{1}{\mu_0} \boldsymbol{B} - \boldsymbol{M} \tag{1.6}$$

However, in linear media, these simplify to $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ and $\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B}$; Where $\varepsilon \equiv \varepsilon_0 \varepsilon_r$, and similar for the permeability.

The Lorentz force law:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{1.7}$$

The Poynting vector:

$$\boldsymbol{P} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B}) \tag{1.8}$$

Not to be confused with the polarisation vector above!

2 Electromagnetic Field Equations

2.1 Maxwell's Equations

Let us start with Coulomb's law: the force between two point charges.
Let us have two charges q_1, q_2, a distance d apart, then, the force between the two is given by:

$$|F_{12}| = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{d^2}$$

The factor of $\frac{1}{4\pi\varepsilon_0}$ is purely due to the SI units system used, and will give the field in units Vm^{-1}. We can calculate the electric field at the site of some test charge q_{test}, due to some charge q. This can be done for both a point charge q, and some continuous body of charge. We consider both below.

2.1.1 Electric Field Due to Point and Continuous Charges

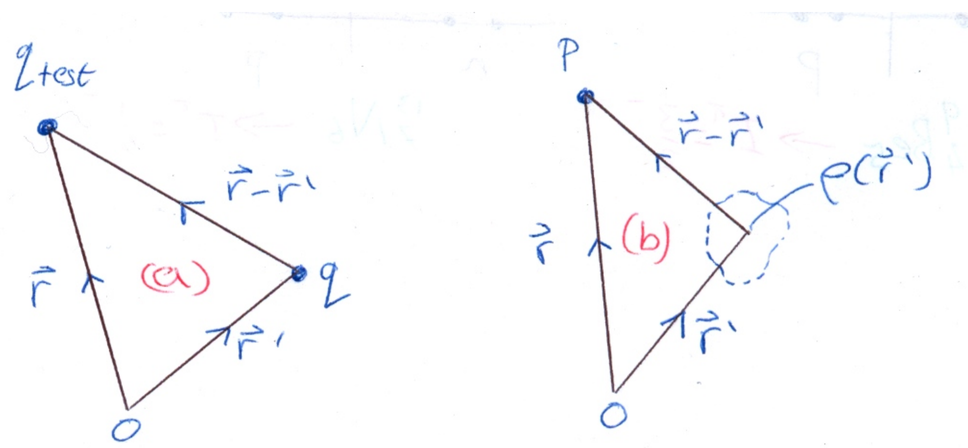

Figure 1: Setup for both point charge (a) and some continuous distribution of charge (b). Note, in both cases, the observation point is at $\boldsymbol{r}$, and the charge of interest is at $\boldsymbol{r}'$.

Consider the setup in Fig (1)(a), for a point charge. The electric field at some position $\boldsymbol{r}$ (that is, at the position of the test charge q_{test}), due to a point charge q, at $\boldsymbol{r}'$, is given by:

$$\begin{aligned} E(\boldsymbol{r}) &= \frac{F}{q_{test}} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{e}_{\boldsymbol{r}-\boldsymbol{r}'} \end{aligned}$$

Where $\boldsymbol{e}_{\boldsymbol{r}-\boldsymbol{r}'}$ is the unit vector between $\boldsymbol{r}$ and $\boldsymbol{r}'$. This is obviously just $\boldsymbol{e}_{\boldsymbol{r}-\boldsymbol{r}'} = \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$, their separation, divided by the magnitude of their separation. Hence,

$$E(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

Now, suppose we have a continuous distribution of charge, as in Fig (1)(b). Let us have a continuous distribution $\rho(\boldsymbol{r}')$. So, our 'total charge' q, which we used before, must now be substituted for an integral over the volume of the charge distribution. Hence, the total charge is $\int \rho(\boldsymbol{r}')\, d^3r'$. Thus, the electric field at the position of the test charge (i.e. at the observation point P) is:

$$E(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{r}')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\, d^3r' \tag{2.1}$$

The integral will sweep over the distribution, picking up the little contributions from each 'bit of charge' (which sounds like a contradiction for a continuous distribution, but its a useful way to think about it).

Now, from vector calculus, we can derive the relation:

$$\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

Hence, we can compare this with (2.1), and substitute in:

$$\begin{aligned} E(\boldsymbol{r}) &= -\frac{1}{4\pi\varepsilon_0}\nabla\int \rho(\boldsymbol{r}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d^3r' \\ &= -\nabla\phi(\boldsymbol{r}) \end{aligned}$$

Where we are now in a position to define the *scalar static potential*:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{r}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d^3r' \tag{2.2}$$

Thus:

$$E(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r}) \tag{2.3}$$

Hence, by Gauss' law (which we will come to shortly), we have that $\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$. Thus, using the above:

$$\nabla\cdot\boldsymbol{E} = -\nabla\cdot\nabla\phi = \frac{\rho}{\varepsilon_0}$$

Which is just:

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \tag{2.4}$$

That is: *Poissons equation.* It is important to note (for subsequent discussions) that the expression $\boldsymbol{E} = -\nabla\phi$ only holds for electrostatic fields. That is, ones for which the charge is at rest. Thus, the above poission equation also only holds for static fields. We shall come to time-varying fields later.

2.1.2 Gauss' Law

Suppose we compute the integral over the (closed) surface containing some volume of charge density:

$$\oint_S \boldsymbol{E}\cdot d\boldsymbol{S} = \frac{1}{\varepsilon_0}\int_V \rho(\boldsymbol{r})\, d^3r \tag{2.5}$$

This is known as the integral version of Gauss' law. By the divergence theorem, we are able to write the LHS surface integral as a volume integral of the divergence of $\boldsymbol{E}$. We can then make the two (arbitrary) volumes on either side the same, leaving us with Maxwell's 1^{st} equation, by equating the integrands:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \tag{2.6}$$

This holds true for time varying and electrostatic fields.

Example Suppose we have an infinite rod, which carries a line-charge density λ (the rod is also infinitely thin). Let us compute the electric field associated with such a system.

Let us choose our surface (i.e we are using the integral version of Gauss' law) to be a cylinder of length ℓ, radius r enclosing the rod. Hence, the LHS of (2.5) is:

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = E_r 2\pi r \ell$$

Where we have noted that all other possible components of the electric field will be parallel to the surface, and will hence not contribute to the surface integral. The RHS of (2.5) is just:

$$\frac{1}{\varepsilon_0} \int_V \rho(\boldsymbol{r})\, d^3 r = \frac{1}{\varepsilon_0} e\lambda\ell$$

Where we have used that the charge per unit length λ will carry a charge of e, hence the total charge along the rod is its charge density multiplied by its length ℓ. Note, we have assigned a length ℓ to an infinite rod, but this isnt too much of a problem, as they cancel. Hence, equating the two sides:

$$\begin{aligned} E_r 2\pi r \ell &= \frac{1}{\varepsilon_0} e\lambda\ell \\ \Rightarrow E_r &= \frac{e\lambda}{2\pi\varepsilon_0 r} \end{aligned}$$

We have hence computed the (radial) electric field at a distance r away from an infinite rod, with charge per unit length λ.
Let us now consider a rod of radius a. We now have:

$$E_r 2\pi r \ell = \frac{1}{\varepsilon_0} \int A \rho_V d\ell$$

Now, the terms in the integral on the RHS: ρ_V is the volume charge density; which is just equal to the line charge density divided by the area in which the charge resides: $\rho_V = \frac{\lambda e}{\pi a^2}$. A is the area over which we are looking: that we have constructed our cylinder: $A = \pi r^2$. Hence:

$$E_r 2\pi r \ell = \frac{1}{\varepsilon_0} \int \pi r^2 \frac{\lambda e}{\pi a^2} d\ell$$

Which is just:

$$E_r 2\pi r \ell = \frac{1}{\varepsilon_0} \frac{r^2}{a^2} \lambda e \ell$$

That is, the field (which is radial) due to a rod of infinite length, but finite radius a is given by:

$$E_r(r) = \frac{1}{\varepsilon_0}\frac{r\lambda e}{2\pi a^2} \qquad r \leq a$$

We shall now discuss the magnetic vector potential, and use it to derive (by analogy) the Biot-Savart law.

2.1.3 Magnetic Vector Potential

Now, from the Maxwell equation which states $\nabla \cdot \boldsymbol{B} = 0$ (i.e. no magnetic monopoles), we are able (via standard vector analysis) to define some vector $\boldsymbol{A}$ that $\boldsymbol{B}$ is the curl of, as the divergence of the curl is zero. That is:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{2.7}$$

So that:

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot \nabla \times \boldsymbol{A} = 0$$

Which, as was previously stated, is satisfied by standard vector analysis; and can be verified easily.

Now, another Maxwell equation (only for static fields) reads:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

Where $\boldsymbol{J}$ is the current density, which is the charge crossing per unit area per unit time. So, inserting our expression for the magnetic vector potential:

$$\nabla \times \nabla \times \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

Which, again by a standard vector identity, can be written:

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

Now, we use what we will call the *Coulomb gauge*, which is the statement that $\nabla \cdot \boldsymbol{A} = 0$. Hence, under the Coulomb gauge, the above becomes:

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J} \tag{2.8}$$

This is the vector form of Poissons equation: i.e. analogous to $\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$.

Magnetic flux is just the surface integral of the magnetic field:

$$\Phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{S}$$

Which, by using the magnetic vector potential, is:

$$\Phi = \int_S \nabla \times \boldsymbol{A} \cdot d\boldsymbol{S}$$

Now, we can apply Stokes theorem to give us:

$$\Phi = \oint_\ell \boldsymbol{A} \cdot d\boldsymbol{\ell} \tag{2.9}$$

So, we have the interpretation that magnetic flux through an area is the same as the line integral of the mangnetic field vector over the line enclosing the surface.

2.1.4 Biot-Savart Law

We shall derive the Biot-Savart law by analogy. Recall that we could find the electric field vector from the scalar potential; so, let us suppose that we could find the magnetic field from the magnetic vector potential. Also, by analogy, let us suppose that the vector potential may be found in the same form as the scalar potential:

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int \boldsymbol{J}\frac{dV}{|\boldsymbol{r}_{12}|}$$

Where $\boldsymbol{r}_{12} \equiv \boldsymbol{r} - \boldsymbol{r}'$ is the vector from the current loop carrying $\boldsymbol{J}$ to the observation point P. The different factor out-front is due to convention in wanting a specific set of units. Now, suppose that

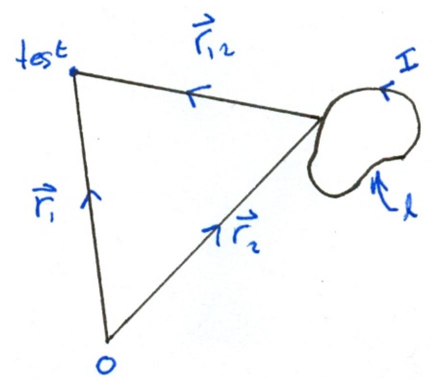

Figure 2: How things are defined in the Biot-Savart law. The integral will sweep over the current loop ℓ, picking up all its contributions.

we consider the loop of current to be a thin wire element, where the wire has cross-section S (which doesn't vary). So, we see that $dV = Sd\ell$. Hence:

$$\boldsymbol{J}dV = \boldsymbol{J}Sd\ell = i_c d\boldsymbol{\ell}$$

Where i_c is the charge flow per unit time. $d\ell$ is the line-element. So now, our integral will sweep over all the conributions from each line element to a field at some observation point. So, we have that:

$$\boldsymbol{A} = \frac{\mu_0 i_c}{4\pi}\oint_\ell \frac{d\boldsymbol{\ell}}{|\boldsymbol{r}_{12}|}$$

Now, we know that we may find $\boldsymbol{B}$ via $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Thus:

$$\boldsymbol{B} = \frac{\mu_0 i_c}{4\pi}\oint_\ell \nabla \times \frac{d\boldsymbol{\ell}}{|\boldsymbol{r}_{12}|}$$

Again, we call upon a vector identity:

$$\nabla \times (a\boldsymbol{v}) = a\nabla \times \boldsymbol{v} + (\nabla a) \times \boldsymbol{v}$$

Where we have identified $|\boldsymbol{r}_{12}|$ as the scalar. Hence:

$$\boldsymbol{B} = \frac{\mu_0 i_c}{4\pi}\left[\oint \frac{1}{|\boldsymbol{r}_{12}|}\nabla \times d\boldsymbol{\ell} + \nabla\left(\frac{1}{|\boldsymbol{r}_{12}|}\right) \times d\boldsymbol{\ell}\right]$$

It is pretty trivial to show that the first term is zero: if one does the cross product, one finds it to be zero. We have also shown that:

$$\nabla\left(\frac{1}{|\boldsymbol{r}_{12}|}\right) = -\frac{\hat{\boldsymbol{r}}_{12}}{|\boldsymbol{r}_{12}|^2}$$

Where $\hat{\boldsymbol{r}}_{12}$ is the unit vector from the current loop to the observation point. We then swop the order of the cross-products, noting that $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$; resulting in:

$$\begin{aligned}\boldsymbol{B} &= \frac{\mu_0 i_c}{4\pi}\oint \frac{d\boldsymbol{\ell}\times\hat{\boldsymbol{r}}_{12}}{|\boldsymbol{r}_{12}|^2} \\ &= \frac{\mu_0 i_c}{4\pi}\oint \frac{d\boldsymbol{\ell}\times\boldsymbol{r}_{12}}{|\boldsymbol{r}_{12}|^3}\end{aligned}$$

Hence, we have the Biot-Savart law:

$$\boldsymbol{B} = \frac{\mu_0 i_c}{4\pi}\oint \frac{d\boldsymbol{\ell}\times\boldsymbol{r}_{12}}{|\boldsymbol{r}_{12}|^3} \tag{2.10}$$

Which finds the magnetic field at a point, from a vector going from the current loop to the observation point. The integral sweeps around the loop, picking up all contributions as it does. We will sometimes use the Biot-Savart law in the following form:

$$d\boldsymbol{B} = \frac{\mu_0 i_c}{4\pi}\frac{d\boldsymbol{\ell}\times\boldsymbol{r}_{12}}{|\boldsymbol{r}_{12}|^3}$$

To be a little more consistent, supose that the current loop is at $\boldsymbol{r}'$ and the observer at $\boldsymbol{r}$ (relative to some origin O), then the Biot-Savart law reads:

$$d\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 i_c}{4\pi}\frac{d\boldsymbol{\ell}\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

2.1.5 Using the Biot-Savart Law

Magnetic Field from a Current Loop Suppose we have a current I going round a loop, of radius r. Suppose that the current is going clock-wise, and that we have a coordinate system centred on $r = 0$, with the $\hat{\boldsymbol{r}}$ direction pointing radially out from the centre, and the $\hat{\boldsymbol{z}}$ direction directly out of the page.
Let us compute the magnetic field at the origin. That is, compute $\boldsymbol{B}(\boldsymbol{0})$; when the current loop is at $\boldsymbol{r}' = r\hat{\boldsymbol{r}}$.

So:

$$
\begin{aligned}
d\boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0 i_c}{4\pi}\frac{d\boldsymbol{\ell}\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \\
\Rightarrow d\boldsymbol{B}(\boldsymbol{0}) &= \frac{\mu_0 I}{4\pi}\frac{d\boldsymbol{\ell}\times(-\boldsymbol{r}')}{|-\boldsymbol{r}'|^3} \\
\Rightarrow d\boldsymbol{\ell}\times(-\boldsymbol{r}') &= |d\boldsymbol{\ell}||-\boldsymbol{r}'|\hat{\boldsymbol{z}} \\
&= d\ell r\hat{\boldsymbol{z}} \\
&= d\ell r\hat{\boldsymbol{z}} \\
\Rightarrow d\boldsymbol{B}(\boldsymbol{0}) &= \frac{\mu_0 I}{4\pi}\frac{d\ell r}{r^3}\hat{\boldsymbol{z}} \\
\Rightarrow \boldsymbol{B}(\boldsymbol{0}) &= \frac{\mu_0 I}{4\pi}\oint_\ell \frac{d\ell}{r^2}\hat{\boldsymbol{z}} \\
&= \frac{\mu_0 I}{4\pi}\frac{2\pi r}{r^2}\hat{\boldsymbol{z}} \\
\Rightarrow B &= \frac{\mu_0 I}{2r}
\end{aligned}
$$

Thus, we have found the magnetic field due to a single loop of wire, carrying current I, at the origin. If the current loop has N (tight) turns, this modifies to:

$$B = \frac{N\mu_0 I}{2r}$$

Magnetic Field due to a Long Straight Wire See Fig (3) to see how we have defined things. So, we have line element $d\boldsymbol{\ell} = dx$.

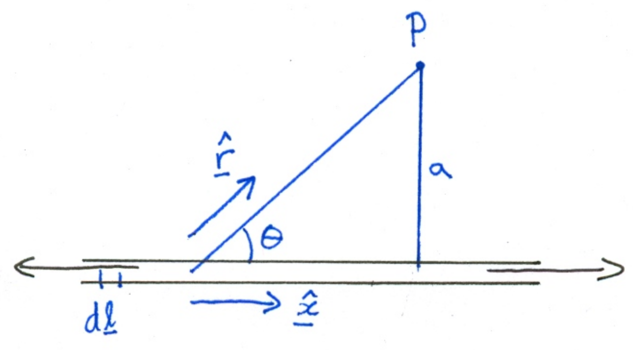

Figure 3: The setup for the magnetic field due to a long thin wire. The wire goes off in either direction to $\pm\infty$. The observer is at P, a vertical distance a from the wire. The wire carries a current I. The vector $\boldsymbol{r}$ goes from the wire to P, along the $\hat{\boldsymbol{r}}$ direction.

$$d\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\frac{d\boldsymbol{\ell}\times\hat{\boldsymbol{r}}}{|\boldsymbol{r}|^2}$$

Let us look at the magnitude of the magnetic field.
So, we first see that:

$$\begin{aligned}|d\boldsymbol{\ell}\times\hat{\boldsymbol{r}}| &= |d\boldsymbol{\ell}||\hat{\boldsymbol{r}}|\sin\theta \\ &= dx\sin\theta\end{aligned}$$

We also see that $|\boldsymbol{r}|^2 = r^2$. Now, by basic trigonometry:

$$r = \frac{a}{\sin\theta}$$

Hence, let us put all this back into the Biot-Savart expression:

$$\begin{aligned}dB &= \frac{\mu_0 I}{4\pi}\frac{|d\boldsymbol{\ell}\times\hat{\boldsymbol{r}}|}{|\boldsymbol{r}|^2} \\ &= \frac{\mu_0 I}{4\pi}\frac{dx\sin\theta\sin^2\theta}{a^2} \\ &= \frac{\mu_0 I}{4\pi}\frac{\sin^3\theta}{a^2}dx\end{aligned}$$

Now, to make this easier to integrate, let us put x in terms of θ. We see that:

$$x = -\frac{a}{\tan\theta} \qquad \Rightarrow dx = \frac{a}{\sin^2\theta}d\theta$$

So, putting this all in:

$$dB = \frac{\mu_0 I}{4\pi a}\sin\theta\, d\theta$$

We must integrate from $\theta = 0 \to \pi$. Hence:

$$B = \frac{\mu_0 I}{4\pi a}\int_0^{\pi} \sin\theta \, d\theta = \frac{\mu_0 I}{2\pi a}$$

Thus, we have found the magnetic field due to a long thin, straight wire, at a distance a from it.

2.1.6 Magnetic Force Between two Parallel Wires

Now, we have just computed that a wire carrying a current generates a magnetic field. We also know that a charge will feel a force in the presence of a magnetic field. Thus, a wire carrying a current will feel a force in a magnetic field.
Thus, two wires carrying a current will exert forces on each other.

Suppose we have two wires, 1 & 2, carrying currents of I_1, I_2 a distance a from each other. Now, the magnetic field a distance a from wire 2 is given by:

$$B_2 = \frac{\mu_0 I_2}{2\pi a}$$

The force felt on a wire, in a magnetic field, if the wire carries a charge q at velocity $\boldsymbol{v}$ is given by:

$$\boldsymbol{F} = q(\boldsymbol{v} \times \boldsymbol{B})$$

Or, in terms of magnitudes $F = qvB$. Now, $qv = I\ell$. Hence, the force on wire 1, due to the magnetic field generated by wire 2 is:

$$F_1 = \frac{\mu I_1 I_2}{2\pi a}\ell_1$$

Where ℓ_i is the length of wire i. Similarly, the force felt on wire 2, due to the field generated by 1 is:

$$F_2 = \frac{\mu I_1 I_2}{2\pi a}\ell_2$$

Which is the same. Hence, the force per unit length is:

$$\frac{F}{\ell} = \frac{\mu I_1 I_2}{2\pi a}$$

So, if the magnitude of the force per unit length between two parallel wires carrying identical currents, and are separated by 1m, is 2×10^{-7}N/m, then the current in each wire is 1A.
This is the definition of the Ampere.

The definition of the Coulomb is similar: If a current of 1A is passing though a wire, then 1C of charge passes a surface in 1s.

2.1.7 Amperes Law

Consider a closed loop ℓ enclosing some surface S; where there is a total current I through the surface S. Then, Amperes law is:

$$\oint_\ell \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 \int_S \boldsymbol{J} \cdot d\boldsymbol{S} \tag{2.11}$$

Using Stoke's theorem, we get:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{2.12}$$

Note, this is only true for static fields. So, consider the following:
Charge conservation is the statement that the rate of flow of current density plus the rate of change of charge density is zero. That is, charge continuity:

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \tag{2.13}$$

Now, as it stands, Amperes law (2.12) does not satisfy the above. So, finding the divergence of (2.12) gives:

$$\nabla \cdot \nabla \times \boldsymbol{B} = \mu_0 \nabla \cdot \boldsymbol{J}$$

The LHS is zero, by vector identities. Hence, we have that $\nabla \cdot \boldsymbol{J} = 0$. Clearly, by (2.13), this is only true if $\frac{\partial \rho}{\partial t} = 0$; which is not generally the case. That is, we have thus far only found a static-equation. Let us consider the time varying equation, which we suppose to be:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{2.14}$$

Taking the divergence of this:

$$\begin{aligned} \nabla \cdot \nabla \times \boldsymbol{B} &= \mu_0 \nabla \cdot \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \nabla \cdot \boldsymbol{E}}{\partial t} \\ \Rightarrow 0 &= \mu_0 \nabla \cdot \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial t} \\ &= \nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} \end{aligned}$$

Where we have used Gauss' law. Hence, our modified time-varying version of Amperes law is consistent with the continuity equation.

2.1.8 Faradays Law

This is the statement that the rate of change of a magnetic field through some surface S generates an electric field (an EMF) in a loop ℓ enclosing the surface. That is:

$$-\frac{\partial}{\partial t} \int_S \boldsymbol{B} \cdot d\boldsymbol{S} = \oint_\ell \boldsymbol{E} \cdot d\boldsymbol{\ell} \tag{2.15}$$

Using Stokes' theorem results in the differential form:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2.16}$$

2.1.9 Summary of Maxwell's Equations in Vacuum

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \quad (2.17)$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad (2.18)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad (2.19)$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \quad (2.20)$$

2.2 Maxwell's Equations in Materials

Here we consider what happens to electric and magnetic fields if we put them in materials.

2.2.1 Dielectrics

This is for materials in electric fields.
We consider the effect of charges induced in a material, when placed in an electric field. Consider that there is a free charge density ρ_{free} and induced charge density ρ_{ind}. Then, the total charge density is just $\rho = \rho_{free} + \rho_{ind}$. Hence, Gauss' law becomes:

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0}(\rho_{free} + \rho_{ind})$$

Note, the induced charges only exist within the material!
If $\rho_{ind} = -\nabla \cdot \boldsymbol{P}$, where $\boldsymbol{P}$ is the polarisation; so that the above easily becomes:

$$\nabla \cdot (\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho_{free}$$

To tidy this up, let us define the *electric displcement vector*:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \quad (2.21)$$

So, we have the form of Gauss' law in materials:

$$\nabla \cdot \boldsymbol{D} = \rho_{free} \quad (2.22)$$

We also say that $\boldsymbol{P} = \varepsilon_0 \chi_E \boldsymbol{E}$; where χ_E is the *electrical susceptibility*, which may be a tensor, depending on the medium. In linear media only, we say that:

$$\boldsymbol{D} = (1 + \chi_E)\varepsilon_0 \boldsymbol{E} = \varepsilon_r \varepsilon_0 \boldsymbol{E}$$

Where ε_r is the relative permitivity for linear media only.

2.2.2 Diamagnetics

This is for materials in a magentic field, such as in a solenoid.

Let us initially consider the non-time-varying version of Amperes law; i.e. that $\frac{\partial \boldsymbol{E}}{\partial t} = 0$. Also, as in dielectrics, we have that the total current density is the sum of free and induced current densities. Thus:

$$\nabla \times \boldsymbol{B} = \mu_0(\boldsymbol{J}_{free} + \boldsymbol{J}_{ind})$$

Also analogously, let $\boldsymbol{J}_{ind} = \nabla \times \boldsymbol{M}$, where $\boldsymbol{M}$ is the magentisation. Hence, the above easily becomes:

$$\nabla \times (\boldsymbol{B} - \mu_0 \boldsymbol{M}) = \mu_0 \boldsymbol{J}_{free}$$

Again, let us clean up the above, by dividing through by μ_0, and collecting terms into:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_{free}$$

Where:

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$$

Which we call the *magnetic intensity*.
Again, we also say that $\boldsymbol{M} = \chi_H \boldsymbol{H}$, where χ_H is the *magnetic susceptibility*. Therefore:

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \mu_0(1 + \chi_H)\boldsymbol{H} = \mu_0 \mu_r \boldsymbol{H}$$

It is fairly easy to put the time-varying component back in, to see that:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_{free} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{2.23}$$

2.2.3 Summary of Maxwell's Equations in Materials

$$\nabla \cdot \boldsymbol{D} = \rho_{free} \tag{2.24}$$
$$\nabla \cdot \boldsymbol{B} = 0 \tag{2.25}$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2.26}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_{free} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{2.27}$$

2.2.4 Fields Across Boundaries

Consider the new Gauss' law, which we can put into integral form easily, using the divergence theorem:

$$\int_S \boldsymbol{D} \cdot d\boldsymbol{S} = \int_V \rho \, dV$$

Where we have dropped the subscript from the charge density.
Now, consider the surface S to be a box; so that it encloses a volume V, half of which is in medium

1, and the other part in medium 2. The normal unit vector $\hat{\boldsymbol{n}}$ points into medium 1. So, the integral on the LHS is just:

$$\int_S \boldsymbol{D}\cdot d\boldsymbol{S} = \boldsymbol{D}_1\cdot\hat{\boldsymbol{n}} - \boldsymbol{D}_2\cdot\hat{\boldsymbol{n}}$$

These then become only components perpendicular to the boundary (that is, parallel to the direction of the normal), which we denote $D_{i\perp}$. Thus, noting that the above is just equal to the surface charge density:

$$D_{1\perp} - D_{2\perp} = \rho_s$$

That is, $\boldsymbol{D}$ is discontinuous at boundaries if $\rho_s \neq 0$; that is, a discontinuity due to free charges.

In a similar way, consider a small loop, crossing the boundary. The work done by the electric field in taking a small charge in the closed loop is zero. That is:

$$\oint \boldsymbol{E}\cdot d\boldsymbol{\ell} = 0$$

We find that:

$$E_{1//} - E_{2//} = 0$$

That is, the electric field is always continuous across boundaries.

2.3 Potentials & Gauge Invariance

Let us consider Faraday's law:

$$\nabla\times\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}$$

Now, we have already introduced the magnetic vector potential $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, so that the above becomes:

$$\nabla\times\boldsymbol{E} = -\frac{\partial}{\partial t}(\nabla\times\boldsymbol{A})$$

Now, we know that curl grad is zero. That is, for some scalar χ, $\nabla\times\nabla\chi = 0$. So, by adding some $\nabla\chi$ onto $\boldsymbol{A}$, we would not 'notice' the change to $\boldsymbol{B}$. So, let us write:

$$\boldsymbol{A}\longmapsto\boldsymbol{A}+\nabla\chi$$

Now, taking a step backwards again, we have (from above) that Faradays law can be written:

$$\nabla\times\left(\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t}\right) = 0 \tag{2.28}$$

So, let us change $\boldsymbol{E}$ so that the above is still valid. So, let us try:

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t} \tag{2.29}$$

So, let us just check that if we choose (2.29), (2.28) is still valid:

$$\begin{aligned}\nabla\times\left(\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t}\right) &= \nabla\times\left(-\nabla\phi-\frac{\partial\boldsymbol{A}}{\partial t}+\frac{\partial\boldsymbol{A}}{\partial t}\right)\\ &= -\nabla\times\nabla\phi\\ &= 0\end{aligned}$$

Thus, our choice in (2.29) has not altered (2.28). Let us see what (2.29) has done to Gauss law:

$$\begin{aligned}\nabla \cdot \boldsymbol{E} &= \nabla \cdot \left(-\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}\right) \\ &= -\nabla^2\phi - \frac{\partial \nabla \cdot \boldsymbol{A}}{\partial t} \\ &= \frac{\rho}{\varepsilon_0} \\ \Rightarrow \nabla^2\phi + \frac{\partial \nabla \cdot \boldsymbol{A}}{\partial t} &= -\frac{\rho}{\varepsilon_0}\end{aligned}$$

This is a coupled wave equation, in both fields ϕ and $\boldsymbol{A}$, driven by charge density ρ.

Now, using Amperes law:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

Where we have used that $c^2 = \frac{1}{\varepsilon_0 \mu_0}$. Noting that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$

$$\nabla \times \nabla \times \boldsymbol{A} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

Using a vector identity:

$$\begin{aligned}\nabla \times \nabla \times \boldsymbol{A} &= \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \\ &= \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}\end{aligned}$$

Now, inserting our choice for $\boldsymbol{E}$ (2.29):

$$\begin{aligned}\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} &= \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}\right) \\ \Rightarrow \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) &= -\mu_0 \boldsymbol{J}\end{aligned}$$

Now, a massive simplification comes when we employ the *Lorentz Gauge*:

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{2.30}$$

Then, we are just left with:

$$\Rightarrow \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}$$

That is:

$$\Rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} = -\mu_0 \boldsymbol{J} \tag{2.31}$$

Which is just a wave equation, which is generated by $\boldsymbol{J}$. Notice that this is decoupled; in contrast to the above driven by ρ.

Basically, as we have that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and $\boldsymbol{E} = -\nabla\phi$, we can change $\boldsymbol{A}$ and ϕ up to the point that they dont change $\boldsymbol{E}, \boldsymbol{B}$. That is, we can make the transformations:

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi \tag{2.32}$$

$$\phi' = \phi - \frac{\partial\chi}{\partial t} \tag{2.33}$$

Where χ is some arbitrary scalar function. Let us now see that we can indeed do this. The criteria we must satisfy is that by changing the potentials, we do not affect the fields.

So, for the magentic field:

$$\begin{aligned}\boldsymbol{B} &= \nabla \times \boldsymbol{A}' \\ &= \nabla \times \boldsymbol{A} + \nabla \times \nabla\chi \\ &= \nabla \times \boldsymbol{A}\end{aligned}$$

Where the final term is zero as the curl of the gradient of a field is zero. Thus, we see that $\boldsymbol{B}$ is invariant under the transformation.

Starting with Faradays law, i.e. $\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}$, and inserting our transformed vector potential, we end up with:

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0$$

Which we have already shown. Now, we know that $\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$. So, let us suppose that we can transform our potentials somewhat:

$$\begin{aligned}\boldsymbol{E} &= -\nabla\phi' - \frac{\partial \boldsymbol{A}'}{\partial t} \\ &= -\nabla\phi' - \frac{\partial}{\partial t}(\boldsymbol{A} + \nabla\chi) \\ &= -\nabla\left(\phi' + \frac{\partial\chi}{\partial t}\right) - \frac{\partial \boldsymbol{A}}{\partial t}\end{aligned}$$

Where, thus far, we have only used the relation between transformed & untransformed vector potential. We shall now find out how the scalar potential changes. Now, let:

$$\phi = \phi' + \frac{\partial\chi}{\partial t}$$

Hence, the transformed potential is:

$$\phi' = \phi - \frac{\partial\chi}{\partial t}$$

So everything is consistent, as we see that the above $\boldsymbol{E}$-field is then unchanged.

We have introduced the *Lorentz gauge*, which is applicable to time-varying fields; whereas the previous Coulomb gauge is only to be used on static fields.

Let us have a small mathematical diversion.

2.4 The Dirac-δ Function & Green Functions

These are very useful for dealing with point charges, or infinitely thin sources.

The Dirac-δ function is defined by:

$$\int_{-\infty}^{\infty} f(x)\delta(x-x')\,dx = f(x') \tag{2.34}$$

Notice, it can be used to pick up a single value of a function. One may think about this in the following way: the integral sweeps over the x-space, but the δ-function will only return a non-zero value when $x = x'$. Hence, it will return that value of any function within the integral.
And has the property that is has unit area:

$$\int_{-\infty}^{\infty} \delta(x-x')\,dx = 1$$

In 3D, we have:

$$\delta^3(\boldsymbol{r}-\boldsymbol{r}') = \delta(x-x')\delta(y-y')\delta(z-z')$$

So, analogously:

$$\int f(\boldsymbol{r})\delta^3(\boldsymbol{r}-\boldsymbol{r}')\,d^3r = f(\boldsymbol{r}') \tag{2.35}$$

Let us consider the delta function, having a function as an argument. Let us start with a simple example; with just a constant inside the (delta) function. Now, let us state (then we shall prove it) the following:

$$\delta(a(x-x')) = \frac{1}{|a|}\delta(x-x')$$

To show this, let us actually evaluate:

$$\int_{-\infty}^{\infty} f(x')\delta(a(x-x'))dx'$$

Let us change variables:

$$y = ax \quad y' = ax'$$

So:

$$\int_{-\infty\times a}^{\infty\times a} f(\tfrac{y'}{a})\delta(y-y')dy'/a$$

Notice, if $a < 0$, then the integral is negative. To see this, replace a with $-a$ everywhere above:

$$\int_{-\infty\times(-a)}^{\infty\times(-a)} f(\tfrac{y'}{-a})\delta(y-y')dy'/(-a) = -\frac{1}{a}\int_{\infty}^{-\infty} f(\tfrac{y'}{-a})\delta(y-y')dy'$$

Now, notice, the effect of having 'minus' an integral is to flip the integration limits. So:

$$-\frac{1}{a}\int_{\infty}^{-\infty} f(\tfrac{y'}{-a})\delta(y-y')dy' = \frac{1}{a}\int_{-\infty}^{\infty} f(\tfrac{y'}{-a})\delta(y-y')dy'$$

So, let us take the modulus of a. Hence:

$$\frac{1}{|a|}\int_{-\infty}^{\infty} f(\tfrac{y'}{a})\delta(y-y')dy' = \frac{1}{|a|}f(\tfrac{y}{a}) = \frac{1}{|a|}f(x)$$

Thus:

$$\int_{-\infty}^{\infty} f(x')\delta(a(x-x'))dx' = \frac{1}{|a|}f(x) \tag{2.36}$$

Now, to consider and actual function:

$$\delta(g(x))$$

So, we need to find the zeros of $g(x)$, as we know that the delta-function is non-zero when its argument is zero. Let them be at x_i, so that we have that $g(x_i) = 0$. Let us now do a Taylor expansion about the zero:

$$g(x_i + \epsilon) = g(x_i) + (x - x_i)g'(x_i) + \dots$$

Note, the first term is zero, by definition. Note, here, a prime denotes derivative with respect to x, wheras above, a prime is a way of distinguishing variables. Hence, near a zero:

$$g(x) \approx (x - x_i)g'(x_i)$$

Near a zero. So, we have that:

$$\delta(g(x)) = \delta(g'(x_i)(x - x_i))$$

Near a zero. This is just like we had before, except we have more than one place the argument of the delta-function is zero. So, we must add the contributions up from all the zeros. Hence:

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \tag{2.37}$$

Where x_i is a zero of the function $g(x)$.

Now, suppose we had a set of charges, where charge q_i resides at some position $\boldsymbol{r}_i$, then, we may write the total charge density as:

$$\rho(\boldsymbol{r}) = \sum_i q_i\delta(\boldsymbol{r} - \boldsymbol{r}'_i)$$

Which can be thought of as a set of 'impluse charges'; which may however, be continuous. Or, as another example, consider that some charges are distributed on an infinitely thin shell, of radius a, and that the distribution conforms to some $\sigma(\theta)$. Then, the charge distribution may be written:

$$\rho(\boldsymbol{r}) = \delta(r - a)\sigma(\theta)$$

2.4.1 Green Functions: Electrostatics

If we wish to solve a differential equation of the form:

$$(L_x u)(x) = f(x)$$

Plus boundary conditions; where L_x is a linear hermitian operator. Then, to do so, we can always write the solution to the DE, $u(x)$ as:

$$u(x) = \int G(x, x')f(x')dx' \tag{2.38}$$

Where the *Green function* is defined by:

$$L_x G(x, x') = \delta(x - x') \tag{2.39}$$

We shall now supress the x-subscript on the operator: it is clear that it only operates on x, not x'.

To prove the above statement isn't too hard:

$$\begin{aligned} (Lu)(x) &= f(x) \\ \Rightarrow L\int G(x, x')f(x')dx' &= \int LG(x, x')f(x')dx' \\ &= \int \delta(x - x')f(x')dx' \\ &= f(x) \end{aligned}$$

Hence proven. In the first line, we used the linearity of the operator, to be able to bring it inside the integral (which is over x' anyway). Then we used the definitions of the Green function and delta function.

Let us proceed by an example from our present electrostatic discussions.

Now, from Gauss' law $\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$ and $\boldsymbol{E} = -\nabla\Phi$, we can easily derive Poissons equation:

$$\nabla^2\Phi(\boldsymbol{x}) = -\frac{\rho(\boldsymbol{x})}{\varepsilon_0}$$

Note, this is strictly for a static charge distribution, else we would have the $\boldsymbol{A}$ term as well. We have the boundary condition that $\Phi(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$.
Now, for a point charge q, at $\boldsymbol{x}'$, we have a charge distribution which is just a delta function:

$$\rho(\boldsymbol{x}) = q\delta(\boldsymbol{x} - \boldsymbol{x}')$$

Note, strictly, we should have written δ^3, but that is understood, as its argument has 3 variables. Now, we know that the resulting Poisson equation has the following solution:

$$\nabla^2\Phi(\boldsymbol{x}) = -\frac{q}{\varepsilon_0}\delta(\boldsymbol{x} - \boldsymbol{x}') \quad \Rightarrow \quad \Phi(\boldsymbol{x}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

Now, upon comparison of the above formalism for Green function, $Lu = f$ is just the Poisson equation. That is, the operatior L is just the ∇^2 operator. Hence, from $LG = \delta(x - x')$, we see that the Green function we want, is that satisfying:

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}')$$

We have the form of G, from the above expression for the point charge. That is:

$$G(\boldsymbol{x}, \boldsymbol{x}') = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

Hence, we have that the solution to Poissons equation is given by:

$$\Phi(\boldsymbol{x}) = \int G(\boldsymbol{x}, \boldsymbol{x}') \left(-\frac{\rho(\boldsymbol{x}')}{\varepsilon_0} \right) d^3x'$$

That is:

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x'$$

Which is something we already knew, but we have derived it using Green functions; and in the process, have identified a Green function.

Notice, we have also arrived at a useful relation:

$$\nabla^2 \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{2.40}$$

Green function theory is extensive, and is used in solving differential equations (as was hinted at previously); but we shall not go into that here.

2.5 Poynting's Theorem

Let us start by stating the Lorentz force law:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

So, the work done dW on a charge dq, when displaced a distance $d\boldsymbol{\ell}$ is given by:

$$dW = \boldsymbol{F} \cdot d\boldsymbol{\ell}$$

That is:

$$dW = dq(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell}$$

Now, from $v = \frac{d}{t}$, a trivial relation, we can write the above as:

$$dW = dq(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v} dt$$

That is:

$$\frac{dW}{dt} = \int dq(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}$$

However, note that:

$$dq = \rho_V d^3r$$

Hence:

$$\begin{aligned} \frac{dW}{dt} &= \int_V \rho_V (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v} \, d^3r \\ &= \int_V \rho_V \left\{ \boldsymbol{E} \cdot \boldsymbol{v} + (\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v} \right\} d^3r \\ &= \int_V \boldsymbol{E} \cdot (\rho_V \boldsymbol{v}) d^3r \end{aligned}$$

Where we have used (the easily verifiable):

$$\boldsymbol{v} \cdot (\boldsymbol{v} \times \boldsymbol{B}) = 0$$

We also note that $\boldsymbol{J} = \rho_V \boldsymbol{v}$. Hence:

$$\frac{dW}{dt} = \int_V \boldsymbol{E} \cdot \boldsymbol{J} \, d^3r \tag{2.41}$$

That is, the total rate of doing work, by the fields, if there is a continuous distribution of charge and current.

Now, consider Amperes' law:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

That is:

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} - \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

So, putting this into (2.41):

$$\begin{aligned} \frac{dW}{dt} &= \int_V \boldsymbol{E} \cdot \left\{ \nabla \times \boldsymbol{H} - \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right\} d^3r \\ &= \int_V \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) - \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} \, d^3r \end{aligned}$$

Now, we use a vector identity for the first expression in the integral:

$$\begin{aligned} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) &= \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) \\ \Rightarrow \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) &= \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) \end{aligned}$$

So, after using this:

$$\frac{dW}{dt} = \int_V \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) - \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} \, d^3r$$

Now, we also know that from Faraday's law:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}$$

Thus, using this:

$$\begin{aligned} \frac{dW}{dt} &= \int_V -\mu_0 \boldsymbol{H} \cdot \frac{\partial \boldsymbol{H}}{\partial t} - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) - \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} \, d^3r \\ &= -\int_V \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) \, d^3r - \mu_0 \int_V \boldsymbol{H} \cdot \frac{\partial \boldsymbol{H}}{\partial t} \, d^3r - \varepsilon_0 \int_V \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} \, d^3r \end{aligned}$$

Now, we can use the divergence theorem on the far LHS integral. That is:

$$\int_V \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) \, d^3r = \int_S (\boldsymbol{E} \times \boldsymbol{H}) \cdot d\boldsymbol{S}$$

Putting this back in:

$$\begin{aligned}\frac{dW}{dt} &= -\int_S (\boldsymbol{E}\times\boldsymbol{H})\cdot d\boldsymbol{S} - \mu_0\int_V \boldsymbol{H}\cdot\frac{\partial\boldsymbol{H}}{\partial t}\,d^3r - \varepsilon_0\int_V \boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\,d^3r \\ &= -\int_S (\boldsymbol{E}\times\boldsymbol{H})\cdot d\boldsymbol{S} - \frac{\partial}{\partial t}\int_V \left\{\varepsilon_0|\boldsymbol{E}|^2 + \mu_0|\boldsymbol{H}|^2\right\}\,d^3r\end{aligned}$$

Now, we start to recognise the *Poynting vector*, and energy densities of the electric and magnetic fields:

$$\boldsymbol{P} = \boldsymbol{E}\times\boldsymbol{H} \tag{2.42}$$

$$U_E = \int_V \varepsilon_0|\boldsymbol{E}|^2\,d^3r \tag{2.43}$$

$$U_M = \int_V \mu_0|\boldsymbol{H}|^2\,d^3r \tag{2.44}$$

Thus, we have:

$$\frac{dW}{dt} = -\frac{\partial}{\partial t}(U_E + U_M) - \int_S \boldsymbol{P}\cdot d\boldsymbol{S} \tag{2.45}$$

So, we see that if the surface integral is zero; the rate at which particles gain energy is equal to the rate at which the fields lose energy.
Also, if the fields are constant (i.e. their time derivatives are zero), and if work is still being done on the particles, then this inflow of energy is provided by the Poynting vector. Thus, we see that the Poynting vector represents the rate at which EM fields transport energy across a unit surface (hence the integral).

As an example, let us consider the energy flux for a plane harmonic wave. That is, ley uscompute the Poynting vector.
So, a plane wave is given by:

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \qquad \boldsymbol{B} = \boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

And, we have that $\boldsymbol{P} = \boldsymbol{E}\times\boldsymbol{H}$. We note that:

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \qquad \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B}$$

So:

$$\frac{\partial\boldsymbol{B}}{\partial t} = -i\omega\boldsymbol{B}$$

And, by doing the cross-product, and realising that the argument of the exponential is actually a scalar, which is $i(k_x x + k_y y + k_z z - \omega t)$; and that $\boldsymbol{E}_0 = (E_x, E_y, E_z)$, where each component has the same exponential factor, we can easily derive:

$$\nabla\times\boldsymbol{E} = i\boldsymbol{k}\times\boldsymbol{E}$$

Thus, we have:

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \quad\Rightarrow\quad i\boldsymbol{k}\times\boldsymbol{E} = i\omega\boldsymbol{B}$$

Hence:

$$\boldsymbol{k} \times \boldsymbol{E} = \omega \mu_0 \boldsymbol{H}$$

Now, we also use the fact that:

$$\hat{\boldsymbol{k}} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$$

Where $\hat{\boldsymbol{k}}$ is a unit vector, in the direction of $\boldsymbol{k}$. We shall write $|\boldsymbol{k}| = k$. We also have the standard relation: $\omega = kc$. So:

$$\begin{aligned}
\boldsymbol{k} \times \boldsymbol{E} &= \omega \mu_0 \boldsymbol{H} \\
\Rightarrow k\hat{\boldsymbol{k}} \times \boldsymbol{E} &= \omega \mu_0 \boldsymbol{H} \\
\Rightarrow \boldsymbol{H} &= \frac{1}{k\omega\mu_0} \hat{\boldsymbol{k}} \times \boldsymbol{E} \\
&= \frac{1}{\mu_0 c} \hat{\boldsymbol{k}} \times \boldsymbol{E}
\end{aligned}$$

Hence, the Poynting vector is:

$$\begin{aligned}
\boldsymbol{P} &= \boldsymbol{E} \times \boldsymbol{H} \\
&= \frac{1}{\mu_0 c} \boldsymbol{E} \times \hat{\boldsymbol{k}} \times \boldsymbol{E} \\
&= \frac{1}{\mu_0 c} |\boldsymbol{E}|^2
\end{aligned}$$

This is, however, an instantaneous value. The time averaged value is just:

$$\langle |\boldsymbol{E}|^2 \rangle = \frac{1}{2} |\boldsymbol{E}|^2$$

2.6 Laplace Equation & Its Solutions

Here, we shall denote the scalar potential as V, rather than ϕ, to avoid any possible confusion with the coordinate.

Now, from Gauss' law, using $\boldsymbol{E} = -\nabla V$, we are able to easily derive the electrostatic Poisson equation:

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Where V is the scalar potential. We have seen that this has solution:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, d^3 r'$$

If there is no charge density, we have the Laplace equation:

$$\nabla^2 V = 0 \tag{2.46}$$

In solving the equation, we use the separation of variables technique; and its solution is unique - up to additive constants.
We can use various boundary conditions:

- Dirichlet: $V(r)$ is known on some surface;
- Neumann: $\frac{\partial V}{\partial n} \equiv \hat{\boldsymbol{n}} \cdot \nabla V$ is known on S;
- Or, a mix of the above two.

We can solve using the inherent symmetry of the system:

- Rectangular - use Cartesian coordinates;
- Cylindrical - use cylindrical polars;
- Spherical - use spherical polars.

We shall now consider various types of solutions.

2.6.1 Solution to the Laplace Equation: Cartesian

This method will apply to all cartesian systems, with different boundary conditions giving different final results. The general method is more-or-less unchanged.

Consider 2 plates which are infinite in z. Let one edge of the plates be held at some potential V_0. So, consider the Dirichlet boundary conditions:

- $V(x, 0) = V(x, a) = 0$;
- $V(0, y) = V_0$;

- $\lim_{x\to\infty} V(x,y) = 0$.

So, we have that the Laplace equation reduces to:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Where the potential $V(x,y)$. So, using separation of variables, we have that:

$$V(x,y) = X(x)Y(y)$$

Putting this into the Laplace equation:

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Dividing through by $V = XY$:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

So, we have that each of the above expressions must be a constant:

$$k^2 + (-k^2) = 0$$

That is:

$$\begin{aligned}\frac{1}{X}\frac{d^2X}{dx^2} &= k^2\\ \frac{1}{Y}\frac{d^2Y}{dy^2} &= -k^2\end{aligned}$$

Each of these is easily solved:

$$X(x) = Ae^{kx} + Be^{-kx} \tag{2.47}$$

$$Y(y) = C\sin ky + D\cos ky \tag{2.48}$$

So, we have that our *general solution* to the Laplace equation, in 2D (or 3D, with symmetry) is:

$$V(x,y) = \{Ae^{kx} + Be^{-kx}\}\{C\sin ky + D\cos ky\} \tag{2.49}$$

Where the constants must be determined by the boundary conditions, and will generally be linear superpositions. So, let us continue, solving for our specific set of boundary conditions:

Let us use the boundary condition that V must decay as $x \to \infty$. That is, $A = 0$. Hence, we have:

$$V(x,y) = e^{-kx}(C\sin ky + D\cos ky)$$

Let us use another boundary condition: $V(x,0) = 0$. Hence, we see that $D = 0$. Hence, we now have:

$$V(x,y) = Ce^{-kx}\sin ky$$

Let us use another boundary condition: $V(x,a) = 0$. So:

$$\sin ka = 0$$

Thus:

$$k = \frac{n\pi}{a} \qquad n = 1, 2, 3, \ldots$$

Therefore:

$$V(x,y) = Ce^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

Using the linear superposition of solutions, we have that:

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right) \tag{2.50}$$

Let us find the constant C_n. Let us use the final boundary condition: $V(0,y) = V_0$. Hence:

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0 \tag{2.51}$$

To go further, we use the orthogonality of sine functions. That is, we use:

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy = \frac{a}{2}\delta_{nm}$$

So, let us multiply both sides of (2.51) with "another", and integrate. That is:

$$\sum_n C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy = V_0 \int_0^a \sin\left(\frac{m\pi y}{a}\right) dy$$

Thus:

$$\sum_n \frac{a}{2} C_n \delta_{nm} = \frac{V_0 a}{m\pi}(1 - \cos m\pi)$$

That is:

$$\frac{a}{2} C_m = \frac{V_0 a}{m\pi}(1 - (-1)^m)$$

So, we see that if m is even, then C_m is zero. And, if m is odd, then:

$$C_m = \frac{4V_0}{m\pi} \qquad \forall m \text{ odd}$$

Therefore, we have:

$$V(x,y) = \sum_{m=1}^{\infty} \frac{4V_0}{m\pi} e^{-\frac{m\pi x}{a}} \sin\left(\frac{m\pi y}{a}\right) \qquad m \text{ odd} \tag{2.52}$$

Thus, we have found the solution of the Laplace equation, in Cartesian coordinates, under our given boundary conditions.

2.6.2 Solution to the Laplace Equation: Cylindrical Polars

Here, we use the coordinate system $V(r, \varphi, z)$; so that the separation of variable happens thus:

$$V(r, \varphi, z) = R(r)\Phi(\varphi)Z(z)$$

So, we have that the Laplacian, in cylindrical polars, takes on the form:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}$$

Thus, using our separated variables:

$$\frac{1}{R}\frac{d^2 R}{dr^2} + \frac{1}{Rr}\frac{dR}{dr} + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\varphi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

Now, we notice a similar linear independence of the terms as for the Cartesian case. Thus, we let:

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = b^2$$

Hence, we have that the Laplace equation becomes:

$$\frac{r^2}{R}\frac{d^2 R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} + b^2 r^2 = 0$$

Now, we also let:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -\alpha^2$$

Hence, the Laplace equation further reduces to:

$$\frac{r^2}{R}\frac{d^2 R}{dr^2} + \frac{r}{R}\frac{dR}{dr} - \alpha^2 + b^2 r^2 = 0$$

Cleaning up:

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(b^2 - \frac{\alpha^2}{r^2}\right)R = 0$$

So, to summarise, we have that the solution to the Laplace equation, in 3D cylindrical polar coordinates, is the product of the solutions to the following equations:

$$\frac{d^2\Phi}{d\varphi^2} = -\alpha^2\Phi \qquad (2.53)$$

$$\frac{d^2 Z}{dz^2} = b^2 Z \qquad (2.54)$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(b^2 - \frac{\alpha^2}{r^2}\right)R = 0 \qquad (2.55)$$

We have that the solutions to (2.53) and (2.54) are just:

$$\Phi(\varphi) = C_\alpha \cos\alpha\varphi + D_\alpha \sin\alpha\varphi \qquad (2.56)$$

$$Z(z) = E_b \cosh bz + F_b \sinh bz \qquad (2.57)$$

And that we note (2.55) is Bessel's equation, which has solutions:

$$R(r) = A_{ab}J_a(br) + B_{ab}N_a(br) \tag{2.58}$$

Where $J_a(br), N_a(br)$ are Bessel functions of the first and second kinds, respectively; and are generally looked up.

So, we have that the general solution to the Laplace equation, in 3D cylindrical polars, is given by:

$$V(r,\varphi,z) = \sum_{a,b}\{A_{ab}J_a(br) + B_{ab}N_a(br)\}\{C_\alpha \cos\alpha\varphi + D_\alpha \sin\alpha\varphi\}\{E_b\cosh bz + F_b\sinh bz\} \tag{2.59}$$

Where the constants must be determined by initial conditions.

2.6.3 Solution to the Laplace Equation: Spherical Polars

We use the coordinate system $V(r,\theta,\varphi)$, where the Laplace equation takes on the form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial V}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial V}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial^2\varphi^2}\right] = 0$$

Now, we separate V thus:

$$V(r,\theta,\varphi) = R(r)Y_{\ell m}(\theta,\varphi)$$

Where $Y_{\ell m}$ are spherical harmonics.
We note that the above Laplacian has the $\hat{L}^2$ operator. That is, the Laplace equation is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial V}{\partial r} - \frac{1}{r^2}\hat{L}^2 = 0$$

Hence, we have that:

$$\frac{1}{r^2}Y_{\ell m}\frac{d}{dr}r^2\frac{dR}{dr} - \frac{1}{r^2}R\hat{L}^2Y_{\ell m} = 0 \tag{2.60}$$

We know that $Y_{\ell m}$ are eigenfunctions of $\hat{L}^2$ thus:

$$\hat{L}^2Y_{\ell m}(\theta,\varphi) = \ell(\ell+1)Y_{\ell m}(\theta,\varphi)$$

Hence, we have that the Laplace equation (2.60) reduces to:

$$\frac{Y_{\ell m}}{r^2}\frac{d}{dr}r^2\frac{dR}{dr} - \frac{R}{r^2}\ell(\ell+1)Y_{\ell m} = 0$$

Now, we try an ansatz: a power law, for $R(r)$. So, let us try $R(r) = r^\alpha$. Hence, putting this into the above yields:

$$\frac{Y_{\ell m}}{r^2}\alpha(\alpha+1)r^\alpha = \frac{r^\alpha}{r^2}\ell(\ell+1)Y_{\ell m}$$

Now, this is just:

$$\alpha^2 + \alpha - \ell - \ell^2 = 0$$

Which we can equivalently write as:

$$(\alpha - \ell)[\alpha + (\ell + 1)] = 0$$

Hence, we see that the two solutions to the above equation, are just:

$$\alpha = \ell \qquad \alpha = -(\ell + 1)$$

And can be verified by substituting back in. So, we have that the general solution to the Laplace equation, in spherical polars is:

$$V(r, \theta, \varphi) = \sum_{\ell, m} \left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \varphi) \tag{2.61}$$

Where we have used both solutions to the ansatz; each having different coefficients $A_{\ell m}, B_{\ell m}$; to be determined by initial conditions.

Now, as $Y_{\ell m}(\theta, \varphi) = \Theta_{\ell}(\theta)\Phi_m(\varphi)$ where $\Phi_m(\varphi) = e^{im\varphi}$, if the system has axial symmetry, then the solution reduces to:

$$V(r, \theta, \varphi) = \sum_{\ell} \left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos\theta)$$

Where the $P_{\ell}(\cos\theta)$ are Legendre polynomials, and can be generated by Rodrigues formula:

$$P_{\ell}(x) = \frac{1}{2^{\ell}\ell!} \left(\frac{d}{dx} \right)^{\ell} (x^2 - 1)^{\ell}$$

And the first few are given by:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\ P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}$$

If the system is symmetrical in both θ, φ; such as a point charge surrounded by dielectric; then the Laplace equation looks like:

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} = 0$$

And the potential is just $V(r) = R(r)$. That is, we must solve:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr} \right) = 0$$

To see why there is no angular components, or differentials; consider that they are in there. That is, the Laplace equation has the form we used previously. The differentials of the angular parts are zero (they are constant: the symmetry of the system), and the radial differential will be multiplied by the angular functions; however, these are then just divided out.
To solve this, note that we must have:

$$r^2 \frac{dR}{dr} = D$$

Where D is some constant. Then the Laplace equation is satisifed. We can find R by the standard method:

$$\begin{aligned} r^2\frac{dR}{dr} &= D \\ \Rightarrow dR &= D\frac{dr}{r^2} \\ \Rightarrow R(r) &= \int D\frac{dr}{r^2} \\ &= -\frac{D}{r} + A \\ &= \frac{B}{r} + A \end{aligned}$$

Where A, B are both constants. Hence, we have that the potential in an angular-symmetric system is given by:

$$V(r) = A + \frac{B}{r}$$

For a more extensive treatise of spherical harmonics, see the relevant appendix.

2.6.4 Example: Dielectric Sphere in Uniform E-field

Consider a dielectric sphere, radius a, relative permittivity ε_r, in a uniform electric field, otherwise vaccum. So, we immediately see that there is axial-symmetry; thus, we solve the Laplace equation in spherical coordinates, with axial symmetry $V(r, \theta)$:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

The boundary conditions are implied, not necessarily given:

- The potential must go to zero at infinity;
- The potential must remain finite (i.e. not diverge) as $r \to 0$;
- The field at infinity is just the 'unperturbed' field, E_0;
- The potential must be continuous at the boundary;
- The perpendicular $D_\perp$ must be continuous at boundaries; as no free charges.

So, as $E = -\nabla V$, we have that $V = -E_0 z$; just the integral. To see the motivation behind this, consider how we choose the alignment of the coordinate system. The alignment is obviously arbitrary, so we choose it to be aligned with the uniform applied field. Remember, the transformation between Cartesian and spherical polars, for z is $z = r\cos\theta$; hence the potential as $r \to \infty$ is just $V = -E_0 r\cos\theta$.
Now, note:

$$P_1(x) = x \quad \Rightarrow \quad P_1(\cos\theta) = \cos\theta$$

So, we have $\ell = 1$ in the summation; by orthogonality considerations of the Legendre polynomials. That is, consider that the potential (given by a sum over an infinite number of multipoles) must go to the dipole term (i.e. $\ell = 1$) at infinity. Thus, multiplying both sides by 'another' polynomial will result in just the dipole term being filtered out, on the potential side. This may be seen in more detail in examples in the appendix. So, taking $\ell = 1$, the 'prototype' for the potential is:

$$V(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta$$

Let us now continue by looking at the two regions (inside and outside) separately, with different constants in both cases. That is:

$$\begin{aligned} V_1 &= \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta \qquad r \leq a \\ V_2 &= \left(A_2 r + \frac{B_2}{r^2}\right)\cos\theta \qquad r > a \end{aligned}$$

Let us consider a boundary condition:

$$\lim_{r\to\infty} V = -E_0 r\cos\theta = V_2$$

Thus, we see that $A_2 = -E_0$; but we can't say anything about B_2.
The consideration that V remain finite at the origin leads us to conclude that $B_1 = 0$. Hence, thus far, we have:

$$\begin{aligned} V_1 &= A_1 r\cos\theta \qquad r \leq a \\ V_2 &= \left(-E_0 r + \frac{B_2}{r^2}\right)\cos\theta \qquad r > a \end{aligned}$$

Let us now consider the first 'boundary' condition; that the potential is continuous at the boundary. That is:

$$V_1(a,\theta) = V_2(a,\theta)$$

So:

$$A_1 a\cos\theta = \left(-E_0 a + \frac{B_2}{a^2}\right)\cos\theta$$

That is:

$$A_1 = \frac{B_2}{a^3} - E_0 \tag{2.62}$$

Let us now apply the second 'boundary' condition; that $D_\perp$ is continuous at the boundary; this is the case because there are no free charges on the boundary. If there were, there is a discontinuity in $D_\perp$, which we consider in later examples. So:

$$D_{1\perp}(r=a) = D_{2\perp}(r=a) \quad \Rightarrow \quad \varepsilon_r E_1(r=a) = E_2(r=a)$$

But, we have that $E = -\frac{\partial V}{\partial r}$; hence:

$$\begin{aligned}
\varepsilon_r \left.\frac{\partial V_1}{\partial r}\right|_{r=a} &= \left.\frac{\partial V_2}{\partial r}\right|_{r=a} \\
\Rightarrow A_1 \varepsilon_r \cos\theta &= \left(-E_0 - \frac{2B_2}{a^3}\right)\cos\theta \\
\Rightarrow A_1 \varepsilon_r &= -E_0 - \frac{2B_2}{a^3}
\end{aligned}$$

So:

$$A_1 \varepsilon_r = -E_0 - \frac{2B_2}{a^3} \tag{2.63}$$

We can easily solve (2.62) and (2.63) for A_1, B_2; giving:

$$A_1 = -\frac{3E_0}{\varepsilon_r + 2} \qquad B_2 = \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) E_0 a^3$$

Hence, putting these back into our potentials for inside and outside:

$$\begin{aligned}
V_1(r,\theta) &= -\frac{3E_0}{\varepsilon_r + 2} r \cos\theta \qquad r \le a \\
V_2(r,\theta) &= -E_0 r \cos\theta + E_0 \cos\theta \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)\frac{a^3}{r^2} \qquad r > a
\end{aligned}$$

Hence solution found.

2.6.5 Example: Charge Inside Spherical Cavity

Consider a charge q at the centre of a spherical cavity; where outside the cavity is a dielectric, permittivity ε_r; and inside vacuum.

Again, the boundary conditions:

- Potential and $D_\perp$ continuous on boundary;
- Potential is zero at infinity;
- Electric field inside cavity, in limit $r \to 0$ is just the Coulomb field.

Here, our system uses spherical polars, but with complete angular symmetry. Hence, the Laplace equation is:

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} = 0$$

We have already discussed the solution to this equation. It is just:

$$V(r) = A + \frac{B}{r}$$

So, again, let us split the problem into inside/outside:

$$\begin{aligned} V_1(r) &= A_1 + \frac{B_1}{r} \qquad r \leq a \\ V_2(r) &= A_2 + \frac{B_2}{r} \qquad r > a \end{aligned}$$

To make the potential zero at $r = \infty$, we consider V_2; and hence conclude that $A_2 = 0$. Let us consider the continuity of the potential at the boundary:

$$A_1 + \frac{B_1}{a} = \frac{B_2}{a}$$

Using $A_2 = 0$. That is:

$$B_2 = aA_1 + B_1$$

Let us consider the continuity of $D_\perp$ at the boundary. Hence, we have:

$$\left.\frac{dV_1}{dr}\right|_{r=a} = \varepsilon_r \left.\frac{dV_2}{dr}\right|_{r=a}$$

Noting that this time the vacuum is on the inside, and dielectic on the outside.
Thus:

$$-\frac{B_1}{a^2} = -\varepsilon_r \frac{B_2}{a^2}$$

Thus:

$$B_1 = \varepsilon_r B_2$$

Now, let us consider $r \to 0$. That is, inside:

$$E_1 \to \frac{q}{4\pi\varepsilon_0 r^2}$$

Noting:

$$E_1 = -\frac{dV_1}{dr} = -\left(-\frac{B_1}{r^2}\right)$$

Hence:

$$B_1 = r^2 E_1 = r^2 \frac{q}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0}$$

We also have, via $B_1 = \varepsilon_r B_2$, that:

$$B_2 = \frac{q}{4\pi\varepsilon_0\varepsilon_r}$$

And, via $B_2 = aA_1 + B_1$, we have that:

$$A_1 = -\frac{q}{4\pi\varepsilon_0 a}\left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right)$$

Therefore, our solution is:

$$\begin{aligned} V_1(r) = -\frac{q}{4\pi\varepsilon_0 a}\left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right) + \frac{q}{4\pi\varepsilon_0 r} \qquad r \leq a \\ V_2(r) = \frac{q}{4\pi\varepsilon_0\varepsilon_r r} \qquad r > a \end{aligned}$$

Let us consider what the surface charge induced is.
Recall that $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$; where $\boldsymbol{P}$ is the polarisation. Also recall, that for a linear medium, $\boldsymbol{D} = \varepsilon_r \varepsilon_0 \boldsymbol{E}$. Hence, after a trivial rearangement:

$$\boldsymbol{P} = (\varepsilon_r - 1)\varepsilon_0 \boldsymbol{E}$$

Let us (as we are liberty to) consider the field just outside the sphere. Then, we have that:

$$\boldsymbol{E}_2 = -\nabla V_2 = \frac{q}{4\pi\varepsilon_0\varepsilon_r r^2}\hat{\boldsymbol{r}}$$

Thus:

$$\boldsymbol{P} = \frac{q(\varepsilon_r - 1)}{4\pi\varepsilon_r r^2}\hat{\boldsymbol{r}}$$

Finally, recall that $\rho_{ind} = -\nabla \cdot \boldsymbol{P}$. Then, we have that:

$$\rho_{ind} = -\frac{q(\varepsilon_r - 1)}{4\pi\varepsilon_r}\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = -\frac{q(\varepsilon_r - 1)}{\varepsilon_r}\delta(r)$$

We have thus computed the surface charge density induced; where we have used:

$$\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = 4\pi\delta(r)$$

2.7 Multipoles

2.7.1 Electric Dipole

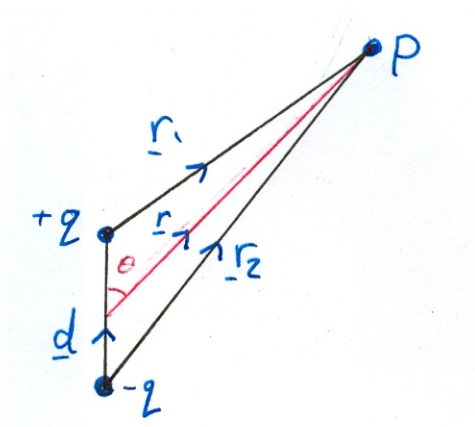

Figure 4: The electric dipole. Two charges, q, $-q$, separated by a distance d, observed at P; which is at a position $\boldsymbol{r}$, as measured from $\frac{d}{2}$

Consider the potential due to two charges, q_1 and q_2, at distances r_1, r_2 from some observation point P. Linear superposition allows us to write that the the 'composite' potential is just:

$$V(P) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right)$$

Now, consider the slightly more specific case, when $q_1 = -q_2 = q$. So:

$$V(P) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Just a quick note, the cosine rule can be very easily derived. Consider:

$$\begin{aligned}(\boldsymbol{r} - \boldsymbol{r}')^2 &= r^2 + (r')^2 - 2\boldsymbol{r}\cdot\boldsymbol{r}' \\ &= r^2 + (r')^2 - 2rr'\cos\theta\end{aligned}$$

So, with reference to the figure:

$$r_1^2 = r^2 + (\tfrac{d}{2})^2 - 2r\tfrac{d}{2}\cos\theta$$

That is:

$$r_1^2 = r^2 + (\tfrac{d}{2})^2 - rd\cos\theta$$

Also, being careful that the angle is now $\cos\alpha = \cos(180 - \theta) = -\cos\theta$, we see that:

$$r_2^2 = r^2 + (\tfrac{d}{2})^2 + rd\cos\theta$$

So, if we combine these results:

$$r_{1,2}^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp rd\cos\theta$$

That is, just:

$$r_{1,2}^2 = r^2\left[1 + \left(\frac{d}{2r}\right)^2 \mp \frac{d}{r}\cos\theta\right]$$

Now, for $r >> d$, we can neglect such quadratic terms. Hence:

$$r_{1,2}^2 = r^2\left[1 \mp \frac{d}{r}\cos\theta\right]$$

Thus, square-rooting:

$$r_{1,2} = r\left[1 \mp \frac{d}{r}\cos\theta\right]^{1/2}$$

Hence:

$$\frac{1}{r_{1,2}} = \frac{1}{r}\left[1 \mp \frac{d}{r}\cos\theta\right]^{-1/2}$$

Now, to first order binomial expansion; i.e. $(1+x)^n \approx 1 + nx$; we have:

$$\frac{1}{r_{1,2}} = \frac{1}{r}\left[1 \pm \frac{d}{2r}\cos\theta\right]$$

Hence:

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{r}\left[1 + \frac{d}{2r}\cos\theta - 1 + \frac{d}{2r}\cos\theta\right] = \frac{d}{r^2}\cos\theta$$

Therefore, we can write the potential, at r, due to a dipole:

$$V(r) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} \tag{2.64}$$

Which we may write as:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2}$$

Where we have defined the *dipole moment* $\boldsymbol{p} \equiv q\boldsymbol{d}$.

2.7.2 General Multipole Expansion

In the present multipole discussion, we assume that there is axial symmetry; which is why we use Legendre polynomials, as will become apparent.

Here, let γ be the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$, as shown in the figure, and $\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{r}'$; so that $\boldsymbol{R}$ is the

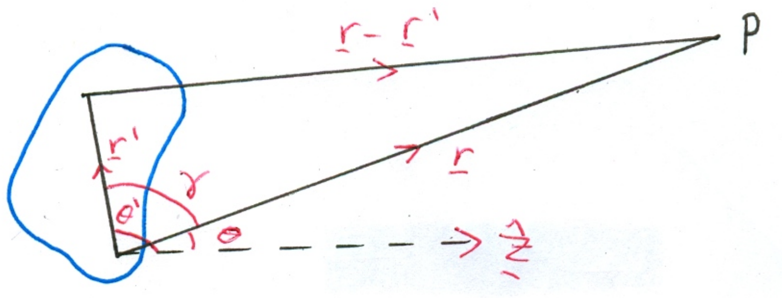

Figure 5: The multipole. Consider a distribution of charges. Notice how things are defined. We have $\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{r}'$, and that $\gamma \equiv \theta' - \theta$; where θ is the angle between the observation point P, and the $\hat{z}$-axis. Similarly, θ' is the angle between the axis and the charge distribution 'bit'.

vector from the charge to the observation point, and γ the angle between charge and observation point. So, under the cosine rule:

$$R^2 = r^2 + (r')^2 - 2rr' \cos\gamma$$

That is:

$$R^2 = r^2 \left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\gamma \right]$$

Now, let:

$$\delta \equiv \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\gamma\right) \tag{2.65}$$

Then:

$$R^2 = r^2(1+\delta) \quad \Rightarrow \quad R = r\sqrt{1+\delta}$$

Hence:

$$\frac{1}{R} = \frac{1}{r}(1+\delta)^{-1/2} \tag{2.66}$$

Now, the binomial expansion is such that:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Therefore:

$$\begin{aligned} (1+\delta)^{-1/2} &= 1 - \frac{1}{2}\delta + \frac{(-1/2)(-3/2)}{2}\delta^2 + \frac{(-1/2)(-3/2)(-5/2)}{6}\delta^3 + \dots \\ &= 1 - \frac{1}{2}\delta + \frac{3}{8}\delta^2 - \frac{5}{16}\delta^3 + \dots \end{aligned}$$

Hence, putting this expansion into (2.66), and taking ourselves out of using δ, via (2.65), we have:

$$\frac{1}{R} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\gamma\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\gamma\right)^2 + \dots\right]$$

If we then gather appropriate terms:

$$\frac{1}{R} = \frac{1}{r}\left[1 + \frac{r'}{r}\cos\gamma + \left(\frac{r'}{r}\right)^2\frac{3\cos^2\gamma - 1}{2} + \left(\frac{r'}{r}\right)^3\frac{5\cos^3\gamma - 3\cos\gamma}{2} + \dots\right]$$

Now, recall the Legendre polynomials:

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{3x^2 - 1}{2} \quad P_3(x) = \frac{5x^3 - 3x}{2}$$

Thus, we notice those terms as present. Hence:

$$\frac{1}{R} = \frac{1}{r}\left[P_0(\cos\gamma) + \left(\frac{r'}{r}\right)P_1(\cos\gamma) + \left(\frac{r'}{r}\right)^3 P_3(\cos\gamma) + \dots\right]$$

We therefore write the whole thing as a sum:

$$\frac{1}{R} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\gamma)$$

Which is just:

$$\frac{1}{R} = \sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}(r')^{\ell}P_\ell(\cos\gamma)$$

Remembering that γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$. The above 'motivation' shows that the expansion seems to work, but is by no means a proof! A proof of this may be found in the relevant appendix, under the Legendre polynomial generating function, and the example relating to our present discussion.

The order to which the sum is taken, is the order of the 'pole'.

So, we have that the potential due to a small bit of charge is just:

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R} \qquad \Rightarrow \qquad V(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{R}\rho(r)d^3r$$

Hence, the total potential, at $\boldsymbol{r}$, due to some charge density $\rho(\boldsymbol{r}')$ is given by:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\int\frac{1}{r^{\ell+1}}P_\ell(\cos\gamma)(r')^{\ell}\rho(\boldsymbol{r}')d^3r'$$

To make this (possibly) a little more transparent, let us write out the first few terms of the sum:

$$\begin{aligned} V(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_0}\int\frac{1}{r}\rho(\boldsymbol{r}')d^3r' + \frac{r'}{r^2}\cos\gamma\rho(\boldsymbol{r}')d^3r' + \frac{r'^2}{2r^3}(3\cos^2\gamma - 1)\rho(\boldsymbol{r}')d^3r' \\ &= \frac{1}{4\pi\varepsilon_0}\int\left[\frac{1}{r} + \frac{r'}{r^2}\cos\gamma + \frac{r'^2}{2r^3}(3\cos^2\gamma - 1)\right]\rho(\boldsymbol{r}')d^3r' \end{aligned}$$

We have that charges are arranged at some $\boldsymbol{r}'$, relative to some orgin, somewhere roughly at the centre of the distribution, and we are looking at the potential at a distance r away from the origin of the system (i.e. at a position $\boldsymbol{r}$). With computations on this, one must be very careful in noting that:

$$\cos\gamma = \cos(\theta' - \theta) = \cos\theta\cos\theta' + \sin\theta\sin\theta'$$

And that the volume element is $d^3r' = r'^2 \sin\theta' dr' d\theta' d\phi'$.

So, we have a multipole expansion of V. We see that the first term is the monopole $\ell = 0$; then dipole $\ell = 1$, quadrupole $\ell = 2$, and continuing up. Notice, at large distances, the multipole expansion will be dominated by the monopole term; or the lowest pole term present.

For a point charge at the origin, all components other than the first are zero; so the monopole is the only contribution:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r}\rho(\boldsymbol{r}')d^3r' = \frac{q}{4\pi\varepsilon_0 r}$$

The total charge is just q.

If the total charge vanishes, then the dipole term is dominant. Now, if we define the *dipole moment* to be:

$$\boldsymbol{p} = \int \boldsymbol{r}'\rho(\boldsymbol{r}')d^3r' \tag{2.67}$$

Then, as $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$, we have:

$$\boldsymbol{p}\cdot\hat{\boldsymbol{r}} = \int \frac{1}{r}\boldsymbol{r}\cdot\boldsymbol{r}'\rho(\boldsymbol{r}')d^3r' = \int \frac{1}{r} r r' \cos\gamma\rho(\boldsymbol{r}')d^3r' = \int r'\cos\gamma\rho(\boldsymbol{r}')d^3r'$$

Where we have noted that the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$ is γ, by definition, with reference to the figure. Which is just the dipole term in the expansion:

$$V = \frac{1}{4\pi\varepsilon_0 r^2}\int r'\cos\gamma\rho(\boldsymbol{r}')d^3r' = \frac{1}{4\pi\varepsilon_0 r^2}\boldsymbol{p}\cdot\hat{\boldsymbol{r}}$$

Recall the dipole moment we defined for a pair of charges: $\boldsymbol{p} = q\boldsymbol{d}$. This is equivalent, as the separation between charges is just r', and total charge is given by the integral over charge density. Thus, the dipole moment is better described in terms of the size, shape & density of the system, as we have with the integral version.

Note, shifting the origin changes multipole moments; however, if the monopole term is zero, the dipole moment is independant of the origin:
Suppose we have a dipole moment relative to some origin 0:

$$\boldsymbol{p} = \int \boldsymbol{r}'\rho(\boldsymbol{r}')d^3r'$$

Then, suppose we shift the coordinates (the origin), so that $\boldsymbol{r}' = \boldsymbol{r}' + \boldsymbol{a}$. Hence, a new dipole moment:

$$\begin{aligned}
\boldsymbol{p}^a &= \int (\boldsymbol{r}' + \boldsymbol{a})\rho(\boldsymbol{r}')d^3r' \\
&= \int \boldsymbol{r}'\rho(\boldsymbol{r}')d^3r' + \int \boldsymbol{a}\rho(\boldsymbol{r}')d^3r' \\
&= \boldsymbol{p} + \boldsymbol{a}q
\end{aligned}$$

But, as we said, $q = 0$; hence, $\boldsymbol{p}^a = \boldsymbol{p}$.

Now, the electric field due to a dipole:

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

Hence, as $\boldsymbol{E} = -\nabla V$, we have its components:

$$\begin{aligned} E_r &= -\frac{\partial V}{\partial r} = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3} \\ E_\theta &= -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3} \\ E_\varphi &= 0 \end{aligned}$$

Hence:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

It is common to notate this as:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}(3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p})$$

So, to conclude, we have that we can write the scalar potential as a sum over multipole moments:

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{p_\ell}{r^{\ell+1}}$$

Where we have the ℓ-pole moment:

$$p_\ell = \int r'^{\ell} P_\ell(\cos\gamma)\rho(\boldsymbol{r}')d^3r'$$

2.7.3 Spherical Harmonic Expansion Multipoles

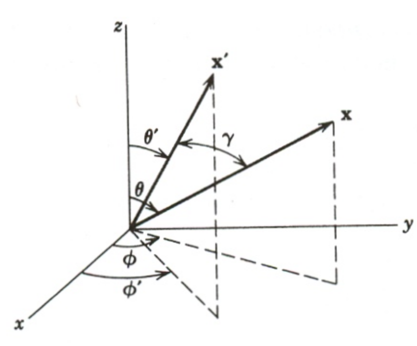

Figure 6: The definition of angles in spherical polars. Notice that the angle between $\boldsymbol{x}$ and $\boldsymbol{x}'$ is γ. Also, we shall be using $\boldsymbol{r}$ as $\boldsymbol{x}$; and $\boldsymbol{r}'$ as $\boldsymbol{x}'$; although this is a trivial assignment of symbols. Figure from Jackson.

Recall the following expansion in terms of Legendre polynomials:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{\ell} \frac{1}{r^{\ell+1}} r'^{\ell} P_{\ell}(\cos\gamma)$$

Where γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$; and has the relation $\boldsymbol{r}\cdot\boldsymbol{r}' = rr'\cos\gamma$. So, we had that the potential may be written:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell}\frac{1}{r^{\ell+1}}\int r'^{\ell}P_{\ell}(\cos\gamma)\rho(\boldsymbol{r}')d^3r' \tag{2.68}$$

Now, we have the following relation:

$$P_{\ell}(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{m=+\ell} Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta',\phi') \tag{2.69}$$

We have that $m = -\ell \to \ell$, in integer steps. Where, from the addition theorem for Spherical harmonics:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

Hence, inserting (2.69) into our potential expression (2.68) above:

$$\begin{aligned} V(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_0}\sum_{\ell}\frac{4\pi}{2\ell+1}\frac{1}{r^{\ell+1}}\sum_{m}\int r'^{\ell}Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta',\phi')\rho(\boldsymbol{r}')d^3r' \\ &= \frac{1}{4\pi\varepsilon_0}\sum_{\ell}\frac{4\pi}{2\ell+1}\frac{1}{r^{\ell+1}}\sum_{m}Y_{\ell m}(\theta,\phi)\int r'^{\ell}Y^*_{\ell m}(\theta',\phi')\rho(\boldsymbol{r}')d^3r' \end{aligned}$$

Now, to tidy this up. Let us write:

$$C_{\ell m}(\theta,\phi) \equiv \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell m}(\theta,\phi) \tag{2.70}$$

So that its conjugate is just (also changing its arguments):

$$C^*_{\ell m}(\theta',\phi') = \sqrt{\frac{4\pi}{2\ell+1}}Y^*_{\ell m}(\theta',\phi')$$

Then, we have that our potential expression cleans up to:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell m}\frac{1}{r^{\ell+1}}C_{\ell m}(\theta,\phi)\int r'^{\ell}C^*_{\ell m}(\theta',\phi')\rho(\boldsymbol{r}')d^3r'$$

Further cleaning this up, let us define the *multipole moment* to be:

$$Q_{\ell m} \equiv \int r'^{\ell}C^*_{\ell m}(\theta',\phi')\rho(\boldsymbol{r}')d^3r' \tag{2.71}$$

Then:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell m}\frac{1}{r^{\ell+1}}C_{\ell m}(\theta,\phi)Q_{\ell m} \tag{2.72}$$

Remember, before, we had:

$$p_\ell = \int r'^\ell P_\ell(\cos\gamma)\rho(\boldsymbol{r}')d^3r'$$

Which is the axially symmetric version of the $Q_{\ell m}$ above.

Now, in rectangular coordinates, we make the 'connection' that:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = e^{-\boldsymbol{r}'\cdot\nabla}\frac{1}{r}$$

So that the expansion is just:

$$e^{-\boldsymbol{r}'\cdot\nabla}\frac{1}{r} = \frac{1}{r} - (\boldsymbol{r}'\cdot\nabla)\frac{1}{r} + \tfrac{1}{2}(\boldsymbol{r}'\cdot\nabla)^2\frac{1}{r} + \ldots + \frac{(-1)^n}{n!}(\boldsymbol{r}'\cdot\nabla)^n\frac{1}{r} + \ldots$$

Where we have used the standard expansion of the exponential:

$$e^x = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n$$

Now, we can evaluate some of the terms in the expansion:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r} + \frac{\boldsymbol{r}'\cdot\boldsymbol{r}}{r^3} + \frac{3(\boldsymbol{r}'\boldsymbol{r})^2 - r'^2r^2}{2r^5} + \ldots$$

Then, using this as the expansion for a multipole:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{1}{2}\sum_{i,j=1}^{3} q_{ij}\frac{x_ix_j}{r^5} + \ldots\right)$$

Where we have used the usual monopole and dipole expressions, as well as the 'new' *quadrupole tensor*:

$$q_{ij} = \int(3x_i'x_j' - r'^2\delta_{ij})\rho(\boldsymbol{r}')d^3r' \qquad (2.73)$$

Which we state, and do not derive.

2.7.4 Relations Between Multipoles in Cartesian & Spherical Polars

Let us compute some components of the multipole:

$$Q_{\ell m} \equiv \int r'^\ell C^*_{\ell m}(\theta',\phi')\rho(\boldsymbol{r}')d^3r'$$

Where the coefficients are given by (infact, its worth noting that they are the associated Legendre polynomials, but we won't go into that here):

$$C^*_{\ell m} = \sqrt{\frac{4\pi}{2\ell+1}}Y^*_{\ell m}(\theta',\phi')$$

Monopole Moment This is the component Q_{00}. So, we look up the spherical harmonic:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \quad \Rightarrow \quad Y^*_{00} = Y_{00}$$

Then, we have:

$$Q_{00} = \int \sqrt{4\pi}\frac{1}{\sqrt{4\pi}}\rho(\boldsymbol{r}')d^3r'$$

Which is just the total charge:

$$Q_{00} = \int \rho(\boldsymbol{r}')d^3r' = q$$

Dipole Moment Let us look up the following spherical harmonics:

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \quad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$

And the final harmonic, i.e. Y_{1-1} can be found from the relation:

$$Y_{\ell-m} = (-1)^m Y^*_{\ell m}$$

Hence, let us start to compute things:

$$\begin{aligned} Q_{10} &= \int r'\sqrt{\frac{4\pi}{3}}\sqrt{\frac{3}{4\pi}}\cos\theta'\rho(\boldsymbol{r}')d^3r' \\ &= \int r'\rho(\boldsymbol{r}')\cos\theta' d^3r' \\ &= p_z \end{aligned}$$

Where we have noted that $r'\cos\theta' = z$, as the standard conversion between spherical polars & cartesian. Thus, we see that Q_{10} is the z-component of the dipole moment.
Next:

$$\begin{aligned} Q_{11} &= \int r'\sqrt{\frac{4\pi}{3}}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta' e^{-i\phi'}\right)\rho(\boldsymbol{r}')d^3r' \\ &= -\frac{1}{\sqrt{2}}\int r'\sin\theta' e^{-i\phi'}\rho(\boldsymbol{r}')d^3r' \\ &= -\frac{1}{\sqrt{2}}\int r'\left[\sin\theta'(\cos\phi' - i\sin\phi')\right]\rho(\boldsymbol{r}')d^3r' \\ &= -\frac{1}{\sqrt{2}}\left[\int r'\sin\theta'\cos\phi'\rho(\boldsymbol{r}')d^3r' - i\int r'\sin\theta'\sin\phi'\rho(\boldsymbol{r}')d^3r'\right] \\ &= -\frac{1}{\sqrt{2}}(p_x - ip_y) \end{aligned}$$

Again, where we have noted the use of the standard polars-cartesian conversion. Finally:

$$\begin{aligned} Q_{1-1} &= \int r'\sqrt{\frac{4\pi}{3}}\left(\sqrt{\frac{3}{8\pi}}\sin\theta' e^{i\phi'}\right)\rho(\boldsymbol{r}')d^3r' \\ &= \frac{1}{\sqrt{2}}\int r'\left[\sin\theta'(\cos\phi' + i\sin\phi')\right]\rho(\boldsymbol{r}')d^3r' \\ &= \frac{1}{\sqrt{2}}(p_x + ip_y) \end{aligned}$$

Doing these, we must be careful that we conjugate the spherical harmonic before putting into the integral.

Quadrupole Moment Here, we compute a component of the quadrupole moment. We look up the following spherical harmonic:

$$Y_{20} = \frac{1}{2}\sqrt{\frac{4}{5\pi}}(3\cos^2\theta - 1)$$

$$\begin{aligned}
Q_{20} &= \frac{1}{2}\int r'^2\sqrt{\frac{4\pi}{5}}\sqrt{\frac{4}{5\pi}}(3\cos^2\theta' - 1)\rho(\boldsymbol{r}')d^3r' \\
&= \frac{1}{2}\int [3(r'\cos\theta')^2 - r'^2]\rho(\boldsymbol{r}')d^3r' \\
&= \frac{1}{2}\int (3z'^2 - r'^2)\rho(\boldsymbol{r}')d^3r' \\
&= \frac{1}{2}q_{33}
\end{aligned}$$

That is, we find that it is related to an element of the quadruple tensor.

2.7.5 Properties of Multipoles

We have see that the details of a charge distribution in a volume, are encoded into the multipoles $Q_{\ell m}$. We have seen that:

- $\ell = 0$ is the monopole moment: $2^0 = 1$ charges;
- $\ell = 1$ is the dipole moment: $2^1 = 2$ charges;
- $\ell = 2$ is the quadrupole moment: $2^2 = 4$ charges;
- $\ell = 3$ is the octupole moment: $2^3 = 8$ charges.

We have also seen that $Q_{\ell m}$ depend on the choice of origin; however, the first non-zero moment is frame independent.
The far field potential is dominated by the first non-zero moment.

2.8 Multipole Expansion of the Vector Potential

As we have previously seen, under the Coulomb gauge, we have that the magentic vector potential satisfies the Poisson equation:

$$\nabla^2\boldsymbol{A} = -\mu_0\boldsymbol{J} \tag{2.74}$$

Which, as we have seen, has solution:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3r' \tag{2.75}$$

Thus, exactly as we did for the scalar potential, we may expand this:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \left[\frac{1}{r} \int \boldsymbol{J}(\boldsymbol{r}') d^3r' + \frac{\boldsymbol{r}}{r^3} \cdot \int \boldsymbol{r}' \boldsymbol{J}(\boldsymbol{r}') d^3r' + \ldots \right]$$

This is a bit more transparent if we consider a single element of the potential:

$$A_i(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \left[\frac{1}{r} \int J_i(\boldsymbol{r}') d^3r' + \frac{\boldsymbol{r}}{r^3} \cdot \int \boldsymbol{r}' J_i(\boldsymbol{r}') d^3r' + \ldots \right] \tag{2.76}$$

Now, looking at the monopole term:

$$\int J_i(\boldsymbol{r}') d^3r' = 0$$

Which is zero (no monopoles). This can also be argued from vector calculus ground; which won't be done here. Looking at the dipole term (the second):

$$\begin{aligned}
\boldsymbol{r} \cdot \int \boldsymbol{r}' J_i(\boldsymbol{r}') d^3r' &= x_j \int x_i' J_i d^3r' \\
&= -\frac{1}{2} x_j \int (x_i' J_j - x_j' J_i) d^3r' \\
&= -\frac{1}{2} \epsilon_{ijk} x_j \int (\boldsymbol{r}' \times \boldsymbol{J})_k d^3r' \\
&= -\frac{1}{2} \left[\boldsymbol{r} \times \int (\boldsymbol{r}' \times \boldsymbol{J}) d^3r' \right]_i
\end{aligned}$$

Hence, we have an expression for the *magnetic dipole moment*:

$$\boldsymbol{m} = \frac{1}{2} \int \boldsymbol{r}' \times \boldsymbol{J}(\boldsymbol{r}') d^3r' \tag{2.77}$$

So that the dipole component of the magnetic vector potential is just (using $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$):

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3} \tag{2.78}$$

Thus, the magnetic field induced by such a vector potential, from $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, is just:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{3(\boldsymbol{m} \cdot \boldsymbol{r})\boldsymbol{r} - r^2 \boldsymbol{m}}{r^5} \tag{2.79}$$

Where we have used the vector identity:

$$\nabla \times (r^n \boldsymbol{a} \times \boldsymbol{r}) = r^{n-2}[(n+2)r^2 \boldsymbol{a} - n(\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{r}]$$

We shall stop with the magnetic analysis here.

2.9 Multipole Expansions Summary

If we have some axially-symmetric charge distribution, at $\boldsymbol{r}'$; with observation point at $\boldsymbol{r}$, then, we are able to expand the distribution in terms of multipoles, so that the scalar potential may be written:

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\int r'^{\ell}P_\ell(\cos\gamma)\rho(\boldsymbol{r}')d^3r' \tag{2.80}$$

Where the angle between the distribution and observation point is $\gamma \equiv \theta' - \theta$.
It can be useful to think about the ℓ-pole term, which will be given by:

$$p_\ell = \int r'^{\ell}P_\ell(\cos\gamma)\rho(\boldsymbol{r}')d^3r' \tag{2.81}$$

So that the potential will be a sum over poles:

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}p_\ell$$

If, however, the charge distribution is not axially-symmetric, then we must appeal to a sum over spherical harmonics, as opposed to Legendre polynomials.

2.10 Method: Potentials and Surface Charges

Suppose we have the problem of finding the scalar potential in all space, in the presence of some symmetric charge distribution.

The first thing to do is to write the relations between $\boldsymbol{D}, \boldsymbol{E}$ and ϕ at the boundary. That is:

$$D_{out} - D_{in} = \rho \qquad \varepsilon_{out}\varepsilon_0 E_{out} - \varepsilon_{in}\varepsilon_0 E_{in} = \rho \qquad \varepsilon_{out}\varepsilon_0\frac{\partial\phi_{out}}{\partial r} - \varepsilon_{in}\varepsilon_0\frac{\partial\phi_{in}}{\partial r} = -\rho \qquad \phi_{out} = \phi_{in}$$

Where *all* are only valid at the boundary. The first expression is that the perpendicular component of the electric displacement field is discontinuous in the presence of surface charge. The second and third expressions just follow from the first, using standard relations to link.

Usually, to find ϕ, the solution in Spherical Polars will be used. This must then be divided into two separate expressions: inside and outside. We must use 'common sense' boundary conditions (it is usual that these are not stated in questions, but must be used). That is, at the origin and infinity, the potential does not diverge. Upon inspection of the potential expressions for inside & outside, this immediately eliminates one coefficient from each expression.
NOTE: this cannot be used if there is an applied electric field. If there is an applied field, then one computes the potential (essentially) at infinity, and uses this as a boundary condition.

Continuing, one is then able to express one coefficient in terms of the other; by using the fact that the potential is continuous at the boundary. To actually do this, one may either use linear independence of the coefficients of Legendre polynomials, and read off the relation directly; or (possibly the more complete method) use orthogonality: multiply the equation by 'another' polynomial.

To then find the remaining coefficient, use the discontinuity of the derivative of the potential, at the boundary. If possible, express the surface charge density in terms of Legendre polynomials, which will make the next step a lot easier to do. Then, once the derivatives done, one must multiply the whole equation by 'another' polynomial, and use orthogonality. Then, one will see that if the surface charge is given by a finite number of polynomials, then the exact forms of the coefficients can be read off. Otherwise, only an approximate 'infinite sum' will result as the coefficients.

Hence, in this way, the scalar field in all space may be found. This method works even if there is a field present, but care must be taken over the 'infinite' boundary condition.

2.11 Discussion

So, let us review this section, and discuss the concepts introduced.

We started by introducing Maxwell's four equations: Gauss' law, no magnetic monopoles, Faradays law & Amperes law. We discussed the electrostatic Coulomb gauge, which allowed us to derive a wave equation for static fields only. We found that we must modify Amperes law, in light of the continuity equation, to take account of time variation of fields.
We then looked a little at the effect of matter & materials on the fields, with boundary conditions for the presence of surface charge, and their effect on the fields.
Next, we looked at time varying fields, employing the Lorentz gauge to derive wave equations which are correct for time-varying fields. We also showed how much freedom we have in choosing the fields, in terms of their associated potentials, in terms of invariance.
We then took a mathematical diversion, looking at the Dirac-δ function, and various properties & uses. As well as a brief note on Green functions, with a specific case given.

We carried on with looking at Poynting's theorem, which gives information on the energy of fields, and the energy flux of a field, over a time period.

Then we discussed the solution to Laplaces equation, which arises in electrostatic systems, where there are regions of no charge density. We discussed its solution in Cartesian, polar & spherical polar coordinates, with solving some special cases with a specific set of boundary conditions.

Finally, we found a way of expressing a continuous distribution of charge, by approximating it to a series of monopoles, dipoles etc. By writing the potentials as a sum like this, we are able to find some properties of the potentials, due to specific distribution of charges. We started with axially symmetric systems (using Legendre polynomials), but continued with removing the axial symmetry (using spherical harmonics: a fuller treatment of spherical harmonics & various related theorems may be found in the appendix).

This concludes our discussion of electrostatic systems. We now move on to considering systems in which charges move, and various consequences of the motion.

3 Retarded Potentials & Radiation

3.1 Introduction to Radiation from Accelerated Charges

Let us begin by considering the derivation of some wave equations. These will be waves 'of potential field', driven by 'charge distributions'.

Consider the following Maxwell's equations, for time-varying fields:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

Consider also:

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

So, inserting the first into Gauss' law:

$$\nabla \cdot \left(-\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} \right) = \frac{\rho}{\varepsilon_0}$$

Giving:

$$\nabla^2 V + \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{A} = -\frac{\rho}{\varepsilon_0} \tag{3.1}$$

And putting both into Amperes law:

$$\nabla \times \nabla \times \boldsymbol{A} = \mu_0 \boldsymbol{J} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} \right)$$

Let us use the following vector identity, and relation:

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \qquad c^2 = \frac{1}{\varepsilon_0 \mu_0}$$

Then, we have:

$$\frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} + \nabla \left(\nabla \cdot \boldsymbol{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right) = \mu_0 \boldsymbol{J} \tag{3.2}$$

Now, if we use the Lorentz gauge:

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0 \quad \Rightarrow \quad \nabla \cdot \boldsymbol{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \tag{3.3}$$

Using this in both (3.1) and (3.2) gives:

$$\frac{1}{c^2} \frac{\partial^2 V}{\partial t} - \nabla^2 V = \frac{\rho}{\varepsilon_0} \tag{3.4}$$

$$\frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J} \tag{3.5}$$

Hence, we have, using the Lorentz gauge, decoupled the equations into two in-homogeneous wave equations. The solutions to these equations are given in terms of retarded scalar and vector potentials.

We must now come to the concept of *retarded time*. If we are at a point in space (i.e. the observer), and we observe a charge distribution, which is moving, then, due to the finite speed of light, we observe the distribution as it *was* when it emitted the light, not how it is now. That is, if we are at a time t, then we see the charge distribution as it was at some time t_{ret}, in the past.

So, we have solutions (which we shall later verify) to the scalar & vector potential:

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}',t_{ret})}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r' \tag{3.6}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}',t_{ret})}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r' \tag{3.7}$$

Where, as usual, the observer is at $\boldsymbol{r}$, and charge distribution of interest (i.e. the retarded distribution) at $\boldsymbol{r}'$. So, one supposes, it would be correct to say $\boldsymbol{r}'(t_{ret})$. That is, $\boldsymbol{r}'$ is the vector from the retarded position to the origin.
Now, as the distance between observer and charge distribution is $|\boldsymbol{r}-\boldsymbol{r}'|$, the the time taken for information to get from the charge distribution (as it was, at t_{ret}), to the observer (at time t) is just $\frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|$. Hence:

$$t_{ret} = t - \frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'| \qquad \left[t_{adv} = t + \frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|\right]$$

Now, the solution to the right, with a +sign, is consistent; but would correspond to an advanced time; which means that the effect would precede the cause. Which is a violation of causality. That is, a potential field *now* would be due to the motion of charges in the *future*. As an aside, it has actually been considered, by Feynman & Wheeler[1] that the potential now is actually a sum of the advanced & retarded potential, and halved. They review a suggestion that radiation could be due to interaction of the radiation, with an observer. The consequence of this is that (and this is an example they give) if we look out of the window, at a star, which is (say) 4 light years away, the only reason it emits radiation, is because the radiation is interacting with an absorber. That is, when the light was emitted (4 years ago), it 'knew' that it would be interacting with material (i.e. the observers eye) 4 year in its future. The cause is after the effect. It is presumably like saying that a tree does not make a sound if it falls in a forest, with no-one around to listen. Anyway, we digress:

Now, suppose we consider point charges:

$$\rho(\boldsymbol{r},t) = q\delta(\boldsymbol{r}-r_0(t)) \quad \boldsymbol{J}(\boldsymbol{r},t) = q\boldsymbol{v}(t)\delta(\boldsymbol{r}-r_0(t))$$

Suppose that the charge distribution is travelling at velocity $\boldsymbol{\beta}c$ (at retarded time). Then, after a fair amount of algebra, which we will do later; we end up with the *Lienard-Wiechart potentials*:

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\kappa R}\right]_{ret} \tag{3.8}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\left[\frac{c\boldsymbol{\beta}}{\kappa R}\right]_{ret} \tag{3.9}$$

[1] *Interaction with the Absorber as the Mechanism of Radiation*: J.Wheeler & R.Feynman Rev.Mod.Phys Vol17, No2, 1945

Where, if $\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{r}'$:

$$\hat{\boldsymbol{R}} = \frac{\boldsymbol{R}}{R} \qquad \kappa = 1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta}$$

So, for velocities close to the speed of light (i.e. $\beta \approx 1$), we see that there will be a strong angle-dependance, due to $\hat{\boldsymbol{R}} \cdot \boldsymbol{\beta} = \beta c \cos\theta$. So, a strong dependance on the angle that the moving charge is viewed, for speeds close to that of c.

So, how does this effect the radiation of energy; i.e. the Poynting vector $\boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{H}$ (which is the energy density flow, per unit area). If a charge is static, then $\boldsymbol{A} = 0$. Hence, $\boldsymbol{P} = 0$; hence, static charges do not radiate.

Now, we know that if a particle is moving at constant velocity, then an inertial frame of reference can be found in which the particle is stationary. Hence, for any charge which is moving at a constant speed, an inertial frame of reference can be found in which it is stationary. Hence, it does not radiate. Thus, any charge moving at a constant velocity does not radiate. Only accelerating charges radiate.

Now, once the potentials are converted into fields, we will find that there are two terms: one dependant upon velocity: $c\boldsymbol{\beta}$, and one dependant upon acceleration: $c\dot{\boldsymbol{\beta}}$. Hence, due to our previous arguement, only the acceleration terms will radiate.

Let us look at the radial dependancies of the energy flux (i.e. of the radiation, $\boldsymbol{P}$). Now, if the maths is done, we find a (electric) 'velocity field', which is proportional to $\frac{1}{R^2}$, whereas the 'acceleration field' is proportional to $\frac{1}{R}$. Now, as magnetic field is proportional to electric field, up to $\boldsymbol{B} = \frac{1}{c}\hat{\boldsymbol{k}} \times \boldsymbol{E}$, we see that for the velocity field, $P \propto \frac{1}{R^2}\frac{1}{R^2} = \frac{1}{R^4}$. Hence, the radiation field, for the velocity field, is $\propto \frac{1}{R^4}$. This is the radiation per unit area, hence, the total radiation flux is the integral $\int P d^2 r \propto \frac{1}{R^2}$. Hence, the radiation field for the velcity field decays to zero.
Now, if we do this for the acceleration field, we see that $P \propto \frac{1}{R^2} \Rightarrow \int P d^2 r \to 1$. That is, the radiation field, for the acceleration field, is a constant.
Hence, because the velocity term decays, but the acceleration term stays constant, we refer to the acceleration term as *the radiation field*.
The equations reffered to are:

$$\boldsymbol{E}_{rad} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{\boldsymbol{R}} \times [(\hat{\boldsymbol{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{c(1 - \boldsymbol{\beta} \cdot \hat{\boldsymbol{R}})^3 R}\right] \qquad \boldsymbol{E}_{velocity} = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\hat{\boldsymbol{R}} - \boldsymbol{\beta})(1 - \beta^2)}{c(1 - \boldsymbol{\beta} \cdot \hat{\boldsymbol{R}})^3 R^2}\right]$$

Both evaluated at the retarded time. Notice, the first term is the acceleration field, which (at the end of our discussion) we call the radiation field. These equations are merely stated here, but will be discussed in detail later.

Consider the following magnitudes, for low β. That is, for non-relativistic motion of charges:

$$\begin{aligned} E_{vel} &= \frac{q}{4\pi\varepsilon_0}\frac{1}{\kappa^3 R^2} \\ E_{rad} &= \frac{q}{4\pi\varepsilon_0}\frac{\dot{\beta}}{c\kappa^3 R} \end{aligned}$$

Where $\kappa \equiv 1 - \boldsymbol{\beta} \cdot \hat{\boldsymbol{R}}$. So, taking the ratio:

$$\frac{E_{rad}}{E_{vel}} = \frac{R}{c}\dot{\beta}$$

From which is see that if no acceleration (i.e. $\dot{\beta} = 0$), then no radiation. Also, if c is infinite, then there would be no radiation.

Also, consider radiation with some characteristic frequency of oscillation, via $c = \nu\lambda$, then we get:

$$\frac{E_{rad}}{E_{vel}} = R\frac{\beta}{\lambda}$$

Hence, we see that the velocity field dominates in the near zone, and radiation in the far zone.

3.1.1 Example: Larmor's Formula

Consider the radiation from a non-relativistic particle. So, we have that $\beta << 1$. Then, we have that, in this limit:

$$E_{rad} = \frac{q}{4\pi\varepsilon_0 c}\frac{\dot{\beta}}{R}\sin\theta$$

Where θ is the angle between the direction of acceleration and the observer. Now, we also have:

$$\boldsymbol{B}_{rad} = \frac{1}{c}\hat{\boldsymbol{R}}_{ret} \times \boldsymbol{E}_{rad}$$

Hence, the Poynting vector:

$$\begin{aligned}
\boldsymbol{P} &= \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} \\
&= \frac{1}{\mu_0 c}\boldsymbol{E}_{rad} \times \hat{\boldsymbol{R}}_{ret} \times \boldsymbol{E}_{rad} \\
&= \frac{1}{\mu_0 c}\left(E_{rad}^2\hat{\boldsymbol{R}}_{ret} - (\boldsymbol{E}_{rad}\cdot\hat{\boldsymbol{R}}_{ret})\boldsymbol{E}_{rad})\right) \\
\Rightarrow |\boldsymbol{P}| = P &= \frac{1}{\mu_0 c}E_{rad}^2
\end{aligned}$$

Where we have used that $\boldsymbol{E}_{rad}$ and $\hat{\boldsymbol{R}}_{ret}$ are perpendicular; as we will see later. So, inserting our expressions in:

$$\begin{aligned}
P &= \frac{1}{\mu_0 c}\frac{q^2}{(4\pi\varepsilon_0 c)^2}\frac{\dot{\beta}^2}{R^2}\sin^2\theta \\
&= \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\dot{\beta}^2}{R^2}\sin^2\theta
\end{aligned}$$

Hence, we have arrived at *Larmor's formula*:

$$P = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\dot{\beta}^2}{R^2}\sin^2\theta \tag{3.10}$$

Now, this is the power radiated per unit area, per unit time. So, the total power radiated, per unit time, is just:

$$\frac{dW}{dt} \equiv \bar{P} = \int PR^2 d\Omega$$

That is:

$$\begin{aligned}
\bar{P} = \frac{dW}{dt} &= \frac{q^2}{16\pi^2\varepsilon_0 c}\dot{\beta}^2 \int \frac{\sin^2\theta}{R^2} R^2 \sin\theta d\theta d\phi \\
&= \Gamma 2\pi \int_{\theta=0}^{\pi} \sin^3\theta d\theta \\
&= \Gamma 2\pi \int_{-1}^{1} 1 - x^2 dx \\
&= \Gamma 2\pi \frac{4}{3} \\
&= \frac{q^2}{6\pi\varepsilon_0 c}\dot{\beta}^2
\end{aligned}$$

We shall denote this expression for the total power radiated, under Larmors assumptions:

$$\bar{P}_L = \frac{q^2}{6\pi\varepsilon_0 c}\dot{\beta}^2 \tag{3.11}$$

Hence, just to check that this satisfies our intuition: as we cannot have negative power, the sign of the charge must be even (i.e. the q^2 term). The power radiated must depend only on the acceleration, and not velocity. Thus, our intuition is satisfied.

Again, even though we have assumed $\beta << 1$, if $c \to \infty$, then power radiated goes to zero. Hence, that c is finite is important. Hence, radiation is a relativistic effect.

Note, in Larmor's formula, we see a $\sin^2\theta$ dependance. That is, no power is radiated along the direction of acceleration.

3.1.2 Retarded Potentials & the Wave Equation

Now, in the previous section, we stated the retarded potentials:

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\boldsymbol{r}', t_{ret})}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3r' \tag{3.12}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}', t_{ret})}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3r' \tag{3.13}$$

Where $t_{ret} = t - \frac{1}{c}R$, with $R \equiv |\boldsymbol{r} - \boldsymbol{r}'|$. That is, the above are:

$$\begin{aligned}
V(\boldsymbol{r}, t) &= \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\boldsymbol{r}', t - \frac{1}{c}R)}{R} d^3r' \\
\boldsymbol{A}(\boldsymbol{r}, t) &= \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}', t - \frac{1}{c}R)}{R} d^3r'
\end{aligned}$$

Now, we wish to show that they do indeed satisfy the wave equations; and we do so for the scalar potential, V:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) V(\boldsymbol{r}, t) = -\frac{\rho(\boldsymbol{r}, t)}{\varepsilon_0}$$

Now, to proceed we compute the following Laplacian:

$$\nabla^2 \frac{\rho}{R} = \rho \nabla^2 \frac{1}{R} + \frac{1}{R}\nabla^2 \rho$$

Now, we also have the result:

$$\nabla^2 \frac{1}{R} = -4\pi\delta(\boldsymbol{R}) = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')$$

Now, as we have that $\rho(\boldsymbol{r}', t-\frac{1}{c}R)$, we know it must be a solution to a wave equation (as that what its solutions look like). Thus, its wave equation is:

$$\nabla^2 \rho = \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2}$$

Hence:

$$\nabla^2 \frac{\rho}{R} = \frac{1}{R}\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} - 4\pi\rho\delta(\boldsymbol{r}-\boldsymbol{r}')$$

Now, the point of computing this was to be able to write:

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\int \nabla^2 \frac{\rho(\boldsymbol{r}', t-\frac{1}{c}R)}{R} d^3r' = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{R}\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} - 4\pi\rho\delta(\boldsymbol{r}-\boldsymbol{r}') d^3r'$$

That is:

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int \frac{\rho(\boldsymbol{r}', t-\frac{1}{c}R)}{R} d^3r' - \frac{1}{\varepsilon_0}\int \delta(\boldsymbol{r}-\boldsymbol{r}')\rho(\boldsymbol{r}', t-\tfrac{1}{c}R) d^3r'$$

Which is just:

$$\nabla^2 V = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\varepsilon_0}\rho(\boldsymbol{r}, t)$$

As we have noted that:

$$\int \delta(\boldsymbol{r}-\boldsymbol{r}')\rho(\boldsymbol{r}', t-\tfrac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|) d^3r' = \rho(\boldsymbol{r}, t)$$

Thus, we have shown that the proposed representation of the retarded scalar potential does indeed satisfy the wave equation.

3.2 Lienard-Wiechert Potentials: Point Charges

Here, we consider the scalar & vector potentials generated by a moving point charge q.

Consider a point charge, having some coordinates $\rho(\boldsymbol{r}', t')$. That is, we consider it at some (retarded) position & time: where it was, when it was. So, we may represent this as a delta-function:

$$\rho(\boldsymbol{r}', t') = q\delta(\boldsymbol{r}' - \boldsymbol{r}_0'(t'))$$

That is, $\boldsymbol{r}_0'(t')$ is some position within the charge. We have used that it is at the retarded time:

$$t' = t - \tfrac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|$$

Now, the potential is given by:

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}',t')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r'$$

So, as one can imagine, integration here is complicated, as the retarded time is a function of the integration variable:

$$V(\boldsymbol{r},t) = \frac{q}{4\pi\varepsilon_0}\int \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\delta(\boldsymbol{r}'-\boldsymbol{r}_0'(t'))d^3r'$$

To go further, we look at the charge distribution again. We re-write it using a delta function for the retarded time:

$$\rho(\boldsymbol{r}',t') = q\int \delta(\tau-t')\delta(\boldsymbol{r}'-\boldsymbol{r}_0'(\tau))\,d\tau \qquad t' \equiv t - \tfrac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|$$

Notice then, this integral will only allow the value $\tau = t'$ in the position delta-function. This integral obviously evaluates to exactly what we had before; the reason for using this will (hopefully) become clearer. Then, if we put this into the potential integral above:

$$V(\boldsymbol{r},t) = \frac{q}{4\pi\varepsilon_0}\int\int \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\delta(\tau-t')\delta(\boldsymbol{r}'-\boldsymbol{r}_0'(\tau))\,d\tau d^3r'$$

Notice, the d^3r' integral will just have the effect of sending $\boldsymbol{r}' \to \boldsymbol{r}_0'(\tau)$. Thus:

$$V(\boldsymbol{r},t) = \frac{q}{4\pi\varepsilon_0}\int \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0'(\tau)|}\delta(\tau-t')d\tau$$

To do this integral, we note that t' is a function of $\boldsymbol{r}$:

$$V(\boldsymbol{r},t) = \frac{q}{4\pi\varepsilon_0}\int \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0'(\tau)|}\delta(\tau-(t-\tfrac{1}{c}|\boldsymbol{r}-\boldsymbol{r}_0'(\tau)|))d\tau$$

Let us denote $\boldsymbol{R}(\tau) \equiv \boldsymbol{r}-\boldsymbol{r}_0'(\tau)$. Then, the above is:

$$V(\boldsymbol{r},t) = \frac{q}{4\pi\varepsilon_0}\int \frac{1}{R(\tau)}\delta(\tau-(t-\tfrac{1}{c}R(\tau)))d\tau$$

Let us continue by noting some dirac-delta theory:

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i}{|f'(x_i)|}$$

Where the function $f(x)$ has zeros at x_i. Thus, in our case, we have $f(\tau) = \tau - t + \frac{1}{c}R(\tau)$. Then:

$$\frac{df}{d\tau} = 1 + \frac{1}{c}\frac{dR}{d\tau}$$

And it has zeros at:

$$\tau = t - \frac{1}{c}R(\tau) \equiv t'$$

Infact, this t' is the same 'class' of coordinate as $\boldsymbol{r}_0'$. So, we shall call it t_0'. Hence, using this:

$$\delta(\tau - (t - \tfrac{1}{c}R(\tau))) = \delta(\tau - t_0')\frac{1}{1 + \frac{1}{c}\frac{dR}{d\tau}\big|_{(\tau=t_0')}}$$

Then, our integral for the potential becomes just:

$$V(\boldsymbol{r}, t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{1 + \frac{1}{c}\frac{dR}{d\tau}\big|_{(\tau=t_0')}}\int \frac{1}{R(\tau)}\delta(\tau - t_0')d\tau$$

Which is easily evaluated to be:

$$V(\boldsymbol{r}, t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{1 + \frac{1}{c}\frac{dR}{d\tau}\big|_{\tau=t_0'}}\frac{1}{R(t_0')}$$

Now, we shall show that:

$$\frac{1}{c}\frac{dR}{d\tau}\bigg|_{\tau=t_0'} = -\left.\frac{\boldsymbol{\beta}\cdot\boldsymbol{R}}{R}\right|_{\tau=t_0'}$$

So, let us use the chain rule:

$$\frac{dR}{d\tau} = \frac{dR}{d\boldsymbol{r}_0'}\cdot\frac{d\boldsymbol{r}_0'}{d\tau}$$

So:

$$\frac{dR}{d\boldsymbol{r}_0'} = \frac{d}{d\boldsymbol{r}_0'}|\boldsymbol{r} - \boldsymbol{r}_0'| = -\frac{\boldsymbol{r} - \boldsymbol{r}_0'}{|\boldsymbol{r} - \boldsymbol{r}_0'|} = -\frac{\boldsymbol{R}}{R}$$

And also, we see that the velocity of the beam is present:

$$\frac{d\boldsymbol{r}_0'}{d\tau} = \boldsymbol{v} = \boldsymbol{\beta}c$$

Hence:

$$\frac{1}{c}\frac{dR}{d\tau} = -\frac{\boldsymbol{R}\cdot\boldsymbol{\beta}}{R} = -\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}$$

Thus shown. Hence, our potential is:

$$V(\boldsymbol{r}, t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{1 - \frac{\boldsymbol{R}\cdot\boldsymbol{\beta}}{R}}\frac{1}{R}$$

Where everything is evaluated at $\tau = t_0'$. Thus, we have *the Lienard-Wiechert scalar potential*:

$$V(\boldsymbol{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})}\right]_{ret} \tag{3.14}$$

By very similar considerations, we can find the vector potential:

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}', t')}{|\boldsymbol{r} - \boldsymbol{r}'|}d^3r'$$

Here, we use that:

$$\boldsymbol{J}(\boldsymbol{r}', t') = c\boldsymbol{\beta}(t')\rho(\boldsymbol{r}', t')$$

Then use the exact same argument, for delta-functions. Giving *the Lienard-Wiechert vector potential*:

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{q\mu_0}{4\pi}\left[\frac{c\boldsymbol{\beta}}{R(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})}\right]_{ret} \tag{3.15}$$

Now, from these potentials, we are able to calculate the fields, from:

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}$$

After some (!!) algebra, one obtains the *the Lienard-Wiechert fields*:

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\hat{\boldsymbol{R}}-\boldsymbol{\beta})(1-\beta^2)}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^3R^2} + \frac{\hat{\boldsymbol{R}}\times((\hat{\boldsymbol{R}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})}{c(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^3R}\right] \tag{3.16}$$

$$\boldsymbol{B} = \frac{\mu_0 qc}{4\pi}\left[\frac{(\boldsymbol{\beta}\times\hat{\boldsymbol{R}})(1-\beta^2)}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^3R^2} + \frac{\dot{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{R}}(\boldsymbol{\beta}\times\hat{\boldsymbol{R}})}{c(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^3R} + \frac{\dot{\boldsymbol{\beta}}\times\hat{\boldsymbol{R}}}{c(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^2R}\right] \tag{3.17}$$

Where:

$$\dot{\boldsymbol{\beta}} \equiv \frac{d}{d\tau}\boldsymbol{\beta}(\tau) \tag{3.18}$$

Notice, we have the accleration & velocity terms in both the electric and magnetic field. The acceleration fields are those with a $\frac{1}{R}$ dependance, and velocity have $\frac{1}{R^2}$. We can also recover a previously known relation:

$$\boldsymbol{B} = \frac{1}{c}\hat{\boldsymbol{R}}_{ret}\times\boldsymbol{E} \tag{3.19}$$

3.2.1 Features of Lienard-Wiechert Potentials

The potentials are relativistically correct - the are Lorentz covariant. That is, shifting reference frames dosent change anything it shouldn't!

For point charges, the magnetic field is always perpendicular to the electric field.

We have seen that we can decompose the fields into velocity and acceleration components:

$$\boldsymbol{E} = \boldsymbol{E}_v + \boldsymbol{E}_v \qquad \boldsymbol{B} = \boldsymbol{B}_v + \boldsymbol{B}_a$$

Where:

$$\boldsymbol{E}_v, \boldsymbol{B}_v \propto \boldsymbol{\beta}, \frac{1}{R^2} \qquad \boldsymbol{E}_a, \boldsymbol{B}_a \propto \dot{\boldsymbol{\beta}}, \frac{1}{R}$$

The energy radiated by a moving charge, per unit time is given by:

$$d\bar{P} = \boldsymbol{P}\cdot\hat{\boldsymbol{R}}dA = \boldsymbol{P}\cdot\hat{\boldsymbol{R}}R^2 d\Omega$$

This goes to:

$$\frac{d\bar{P}}{d\Omega} = \lim_{R\to\infty}\left[\boldsymbol{P}\cdot\hat{\boldsymbol{R}}R^2\frac{dt}{dt_{ret}}\right] \tag{3.20}$$

As the energy radiated, per unit time, per unit solid angle.

The components of the Poynting vector are:

$$P_{vv} \propto |E_v \times B_v| \propto \frac{1}{R^4} \tag{3.21}$$

$$P_{va} \propto |E_v \times B_a| \propto \frac{1}{R^3} \tag{3.22}$$

$$P_{aa} \propto |E_a \times B_a| \propto \frac{1}{R^2} \tag{3.23}$$

And thus, the only non-vanishing component, as $R \to \infty$ is P_{aa}.

One may think of the lack-of radiation of the velocity components, as the energy being 'convected' along with the beam. The velocity term does have an associated energy - it is linked to the kinetic energy of the beam.

Let us consider an example.

3.2.2 Example: Particle Moving With Constant Velocity

So, we have a particle moving with constant velocity $c\boldsymbol{\beta}$.

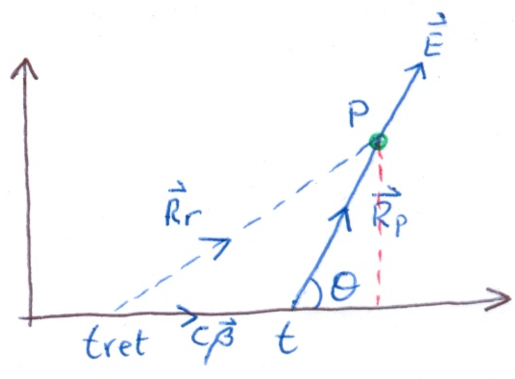

Figure 7: The setup for retarded motion.

If we have that the particle moves with constant velocity, then obviously $\dot{\boldsymbol{\beta}} = 0$. Thus, only the velocity term of the Lienard-Wiechert field contributes:

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\hat{\boldsymbol{R}} - \boldsymbol{\beta})(1-\beta^2)}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^3 R^2}\right]_{ret}$$

With reference to the figure, positions at the retarded time are denoted with a subscript r. Thus, the above is just:

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\frac{(1-\beta^2)}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}}_r)^3 R_r^2}(\hat{\boldsymbol{R}}_r - \boldsymbol{\beta}) \tag{3.24}$$

Now, the main goal is to get everything in terms of positions *now*, rather than at the retarded time. We do this by careful vector algebra.

Consider the distance the particle travels between t_{ret} and t. This is just its velocity multiplied by the difference in times. That is, the distance the particle travels is just:

$$c\boldsymbol{\beta}(t - t_{ret})$$

Hence, with reference to the figure, we may then write:

$$\boldsymbol{R}_r = c\boldsymbol{\beta}(t - t_{ret}) + \boldsymbol{R}_p \tag{3.25}$$

Which is (obviously) completely the same as writing:

$$\boldsymbol{R}_p = \boldsymbol{R}_r - c\boldsymbol{\beta}(t - t_{ret})$$

Also, the distance a signal travel, at speed c, between t_{ret} and t is just R_r; by definition. Thus, we also have:

$$R_r = c(t - t_{ret})$$

Thus, combining the above two results:

$$\boldsymbol{R}_p = \boldsymbol{R}_r - R_r\boldsymbol{\beta}$$

That is, using the standard $\boldsymbol{R} = R\hat{\boldsymbol{R}}$, just:

$$\boldsymbol{R}_p = R_r(\hat{\boldsymbol{R}}_r - \boldsymbol{\beta}) \quad \Rightarrow \quad (\hat{\boldsymbol{R}}_r - \boldsymbol{\beta}) = \frac{\boldsymbol{R}_p}{R_r} \tag{3.26}$$

Hence, using this in (3.24) gives:

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\frac{(1-\beta^2)}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}}_r)^3 R_r^3}\boldsymbol{R}_p \tag{3.27}$$

Thus, the only 'vector' left (and hence direction), is actually the actual position $\boldsymbol{R}_p$, rather than the retarded position $\boldsymbol{R}_r$. Hence, the electric field is in the direction of where the field is now, but due to where it was! So, let us continue, and get the denominator in terms of 'now' variables.

Now, let us square the left-hand expression in (3.26):

$$\begin{aligned} R_p^2 &= R_r^2(\hat{\boldsymbol{R}}_r - \boldsymbol{\beta})^2 \\ &= R_r^2(1 - 2\hat{\boldsymbol{R}}_r\cdot\boldsymbol{\beta} + \beta^2) \\ &= R_r^2 - 2R_r^2\hat{\boldsymbol{R}}_r\cdot\boldsymbol{\beta} + R_r^2\beta^2 \end{aligned}$$

That is:

$$R_p^2 = R_r^2 - 2R_r^2\hat{\boldsymbol{R}}_r\cdot\boldsymbol{\beta} + R_r^2\beta^2 \tag{3.28}$$

Now, with reference to the figure; we notice that the vertical in the two triangles is the same. So:

$$|\boldsymbol{R}_r \times \boldsymbol{\beta}|^2 = |\boldsymbol{R}_p \times \boldsymbol{\beta}|^2 \tag{3.29}$$

Let us expand the LHS, carefully:

$$\begin{aligned}|\boldsymbol{R}_r \times \boldsymbol{\beta}|^2 &= [R_r \beta \sin\theta_r]^2 \\ &= R_r^2 \beta^2 \sin^2\theta_r \\ &= R_r^2\beta^2(1-\cos^2\theta_r) \\ &= R_r^2\beta^2 - (\boldsymbol{R}_r \cdot \boldsymbol{\beta})^2\end{aligned}$$

Where θ_r is the angle between $\boldsymbol{\beta}$ and $\boldsymbol{R}_r$. We have noted that the defintion of the dot-product comes out, which we have then used. We can then easily see that (3.29) is just:

$$R_r^2\beta^2 - (\boldsymbol{R}_r \cdot \boldsymbol{\beta})^2 = R_p^2\beta^2 - (\boldsymbol{R}_p \cdot \boldsymbol{\beta})^2 \tag{3.30}$$

Now, let us compute the following:

$$R_r^2(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^2 = (R_r - \boldsymbol{\beta}\cdot\boldsymbol{R}_r)^2$$

Which gives:

$$R_r^2(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^2 = R_r^2 - 2R_r\boldsymbol{\beta}\cdot\boldsymbol{R}_r + (\boldsymbol{\beta}\cdot\boldsymbol{R}_r)^2$$

Now, substitute (3.28) into the above, for R_r^2; giving:

$$R_r^2(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^2 = R_p^2 + 2R_r^2\hat{\boldsymbol{R}}_r\cdot\boldsymbol{\beta} - R_r^2\beta^2 - 2R_r\boldsymbol{\beta}\cdot\boldsymbol{R}_r + (\boldsymbol{\beta}\cdot\boldsymbol{R}_r)^2 \tag{3.31}$$

If we rearrange (3.30) slightly, we have:

$$(\boldsymbol{R}_r \cdot \boldsymbol{\beta})^2 - R_r^2\beta^2 = (\boldsymbol{R}_p \cdot \boldsymbol{\beta})^2 - R_p^2\beta^2$$

Substituting this into (3.31), for the third and last terms:

$$R_r^2(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^2 = R_p^2 + 2R_r^2\hat{\boldsymbol{R}}_r\cdot\boldsymbol{\beta} - 2R_r\boldsymbol{\beta}\cdot\boldsymbol{R}_r + (\boldsymbol{R}_p\cdot\boldsymbol{\beta})^2 - R_p^2\beta^2$$

Noting that two terms (note that $2R_r^2\hat{\boldsymbol{R}}_r\cdot\boldsymbol{\beta} = 2R_r\boldsymbol{R}_r\cdot\boldsymbol{\beta}$) cancel gives:

$$R_r^2(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^2 = R_p^2 + (\boldsymbol{R}_p\cdot\boldsymbol{\beta})^2 - R_p^2\beta^2$$

Now, notice, the LHS of the above is 'almost' the same factor as we have in the denominator of the electric-field expression (3.27) (it is the same, if we take the 3/2 power of the above). And also, the above is purely in terms of 'now'! Let us expand this out now, getting some angular dependance.

$$\begin{aligned}R_r^2(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^2 &= R_p^2 + (R_p\beta\cos\theta)^2 - R_p^2\beta^2 \\ &= R_p^2 + R_p^2\beta^2(\cos^2\theta - 1) \\ &= R_p^2 - R_p^2\beta^2\sin^2\theta \\ &= R_p^2(1-\beta^2\sin^2\theta)\end{aligned}$$

Hence, we can use this in the denominator of (3.27):

$$\begin{aligned}\boldsymbol{E} &= \frac{q}{4\pi\varepsilon_0}\frac{(1-\beta^2)}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}}_r)^3R_r^3}\boldsymbol{R}_p \\ &= \frac{q}{4\pi\varepsilon_0}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}R_p^3}\boldsymbol{R}_p \\ &= \frac{q}{4\pi\varepsilon_0}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}R_p^2}\hat{\boldsymbol{R}}_p\end{aligned}$$

Thus, we have computed the electric field in terms of the present position of the particle, and the angle in which the particle is viewed:

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0} \frac{(1-\beta^2)}{(1-\beta^2 \sin^2\theta)^{3/2} R_p^2} \hat{\boldsymbol{R}}_p$$

Also, we can find the magnetic field, from the following expression:

$$\boldsymbol{B} = \frac{1}{c} \hat{\boldsymbol{R}}_r \times \boldsymbol{E}$$

Now, from (3.25), we have that:

$$\boldsymbol{R}_r = c\boldsymbol{\beta}(t - t_{ret}) + \boldsymbol{R}_p$$

And hence:

$$\hat{\boldsymbol{R}}_r = \frac{\boldsymbol{R}_r}{R_r} = \frac{c\boldsymbol{\beta}(t - t_{ret})}{R_r} + \frac{\boldsymbol{R}_p}{R_r}$$

We also had that $R_r = c(t - t_{ret})$. Hence:

$$\hat{\boldsymbol{R}}_r = \boldsymbol{\beta} + \frac{\boldsymbol{R}_p}{R_r}$$

And therefore:

$$\boldsymbol{B} = \frac{1}{c}\left(\boldsymbol{\beta} + \frac{\boldsymbol{R}_p}{R_r}\right) \times \boldsymbol{E}$$

Now, $\boldsymbol{R}_p$ and $\boldsymbol{E}$ are along the same line. Hence, that term is zero. Thus:

$$\boldsymbol{B} = \frac{1}{c}\boldsymbol{\beta} \times \boldsymbol{E}$$

Hence, using our electric field:

$$\boldsymbol{B} = \frac{q}{4\pi\varepsilon_0 c} \frac{(1-\beta^2)}{(1-\beta^2 \sin^2\theta)^{3/2} R_p^2} (\boldsymbol{\beta} \times \hat{\boldsymbol{R}}_p)$$

Also, note that:

$$\frac{1}{4\pi\varepsilon_0 c} = \frac{\mu_0 c}{4\pi}$$

Hence, to summarise what we have done: we have found the electric and magnetic fields due to a moving particle. We have found that they do not depend upon retarded time, but only on the time that the particle is in presently.

3.3 Radiation

Let us consider the radiation emitted by a particle moving. We shall consider the two cases of velcity and acceleration being parallel and perpendicular; but before then, we shall derive the general form of radiation from a moving charge.

3.3.1 General Theory of Radiation

Now, we know that radiation is only emitted from the acceleration component of the electric field:

$$\boldsymbol{E}_e = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{\boldsymbol{R}} \times (\hat{\boldsymbol{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}}{(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})R}\right]_{ret} \tag{3.32}$$

With the corresponding (acceleration components of) magnetic field being:

$$\boldsymbol{B}_a = \frac{1}{c}[\hat{\boldsymbol{R}}]_{ret} \times \boldsymbol{E}_a \tag{3.33}$$

Now, from these expressions, we can see that both $\boldsymbol{E}_a$ and $\boldsymbol{B}_a$ are perpendicular to $\hat{\boldsymbol{R}}$. That is:

$$\boldsymbol{E}_a \cdot \hat{\boldsymbol{R}} = 0 \tag{3.34}$$

Now, the radiation per unit area is given by the Poynting vector, as we know:

$$\boldsymbol{P} = \frac{1}{\mu_0}\boldsymbol{E}_a \times \boldsymbol{B}_a$$

Thus, using (3.33) in the above expression:

$$\boldsymbol{P} = \frac{1}{\mu_0 c}\boldsymbol{E}_a \times [\hat{\boldsymbol{R}}]_{ret} \times \boldsymbol{E}_a$$

Let us use the vector identity $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$; which gives:

$$\boldsymbol{P} = \frac{1}{\mu_0 c}\left(E_a^2[\hat{\boldsymbol{R}}]_{ret} - (\boldsymbol{E}_a \cdot [\hat{\boldsymbol{R}}]_{ret})\boldsymbol{E}_a\right)$$

However, from (3.34), this is just:

$$\boldsymbol{P} = \frac{1}{\mu_0 c}E_a^2[\hat{\boldsymbol{R}}]_{ret} \tag{3.35}$$

Now, the amount of energy dW, going through some bit of surface $d\boldsymbol{A}$, per unit time is:

$$\frac{dW}{dt} = \boldsymbol{P}\cdot d\boldsymbol{A} = [\boldsymbol{P}\cdot\hat{\boldsymbol{R}}R^2 d\Omega]_{ret}$$

Now, as the power $\bar{P}$ is $d\bar{P} = \boldsymbol{P}\cdot\hat{\boldsymbol{R}}R^2 d\Omega$ - the amount of 'poynting vector' going through an area, we obviously have:

$$\frac{d\bar{P}}{d\Omega} = [\boldsymbol{P}\cdot\hat{\boldsymbol{R}}R^2]_{ret}$$

Hence, upon comparison of the two above expressions, we can write:

$$\frac{d^2W}{dtd\Omega} = [\boldsymbol{P} \cdot \hat{\boldsymbol{R}} R^2]_{ret}$$

Now, if the charge is accelerated between times $t_1 \to t_2$, how much energy is lost? We can obviously write:

$$\frac{dW}{d\Omega} = \int_{t_1}^{t_2} [\boldsymbol{P} \cdot \hat{\boldsymbol{R}} R^2]_{ret}\, dt$$

Hence, as we can convert between retarded times thus:

$$t_i = t'_i + \frac{R(t'_i)}{c} \qquad \Rightarrow \qquad \frac{dt}{dt'} = 1 + \frac{1}{c}\frac{dR}{dt'}$$

Where we have previously seen that:

$$\frac{1}{c}\frac{dR}{dt'} = -[\hat{\boldsymbol{R}} \cdot \boldsymbol{\beta}]_{ret}$$

Hence, let us change integration variables:

$$\begin{aligned} \frac{dW}{d\Omega} &= \int_{t'_1}^{t'_2} [\boldsymbol{P} \cdot \hat{\boldsymbol{R}} R^2]_{ret} \frac{dt}{dt'} dt' \\ &= \int_{t'_1}^{t'_2} [\boldsymbol{P} \cdot \hat{\boldsymbol{R}} R^2]_{ret} \left(1 - [\hat{\boldsymbol{R}} \cdot \boldsymbol{\beta}]_{ret}\right) dt' \end{aligned}$$

Therefore, divinding out the dt':

$$\frac{d^2W}{dt'd\Omega} = \left[(\boldsymbol{P} \cdot \hat{\boldsymbol{R}} R^2)(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})\right]_{ret}$$

And this, upon comparison of the LHS and previous expressions, is:

$$\frac{d\bar{P}}{d\Omega} = \left[(\boldsymbol{P} \cdot \hat{\boldsymbol{R}})(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})R^2\right]_{ret}$$

Hence, using (3.35) and (3.32), this is just:

$$\frac{d\bar{P}}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c} \left[\frac{\left(\hat{\boldsymbol{R}} \times (\hat{\boldsymbol{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right)^2}{(1 - \boldsymbol{\beta} \cdot \hat{\boldsymbol{R}})^5}\right]_{ret} \tag{3.36}$$

Now, we derived $\bar{P}$ in the comoving frame of the charge q; but the result is valid for any frame, including the rest frame of the lab. Thus, $\bar{P}$ is invariant under Lorentz transformations.

Let us now consider colinear acceleration:

3.3.2 Radiation: Acceleration & Velocity Parallel

This is sometimes denoted *colinear motion*.

Here, we have that the acceleration and velocity of the particle are in the same direction. Hence, this immediately tells us that:

$$\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$$

As is the condition for two things being parallel. Hence, using this:

$$\frac{d\bar{P}}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\left[\frac{\left(\hat{\boldsymbol{R}} \times \hat{\boldsymbol{R}} \times \dot{\boldsymbol{\beta}}\right)^2}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^5}\right]_{ret}$$

Now, we use a vector identity[2], to see that:

$$\hat{\boldsymbol{R}} \times \hat{\boldsymbol{R}} \times \dot{\boldsymbol{\beta}} = \dot{\beta}\cos\theta\hat{\boldsymbol{R}} - \dot{\boldsymbol{\beta}}$$

Thus, squaring it:

$$\begin{aligned}
(\hat{\boldsymbol{R}} \times \hat{\boldsymbol{R}} \times \dot{\boldsymbol{\beta}})^2 &= (\dot{\beta}\cos\theta\hat{\boldsymbol{R}} - \dot{\boldsymbol{\beta}})^2 \\
&= \dot{\beta}^2\cos^2\theta + \dot{\beta}^2 - 2\dot{\beta}^2\cos^2\theta \\
&= \dot{\beta}^2(1-\cos^2\theta) \\
&= \dot{\beta}^2\sin^2\theta
\end{aligned}$$

Hence, using this:

$$\frac{d\bar{P}}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\left[\frac{\dot{\beta}^2\sin^2\theta}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^5}\right]_{ret} \tag{3.37}$$

And, with $\beta << 1$, we obtain the previously derived *Larmor formula*:

$$\frac{d\bar{P}}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\dot{\beta}^2\sin^2\theta \tag{3.38}$$

Where θ is the angle between $\dot{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{R}}$.

Hence, we have derived the angular distribution of radiation, from a charge moving parallel to the direction in which it is being accelerated. We have also derived the form for non-relativistic motion.

Using (3.37), let us find the angle at which most radiation is emmited.
To do this, let us isolate the θ-dependance, and differentiate:

$$\frac{d}{d\theta}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} = \frac{2\cos\theta\sin\theta}{(1-\beta\cos\theta)^5} - \frac{5\beta\sin^3\theta}{(1-\beta\cos\theta)^6} = 0$$

And set it to zero (maximum). Solving:

$$[2\cos\theta(1-\beta\cos\theta) - 5\beta\sin^2\theta]\sin\theta = 0$$

[2] $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$

Solve this by the quadratic formula, with $x \equiv \cos\theta$, giving:

$$x = \frac{-1 \pm \sqrt{1+15\beta}}{3\beta}$$

And thus, the maximum (as opposed to the minimum) is, from $x = \cos\theta$:

$$\theta_{max} = \cos^{-1}\left(\frac{-1+\sqrt{1+15\beta}}{3\beta}\right)$$

Let us finally figure out the total power radiated by a non-relativistic particle. We hence integrate (3.38):

$$\bar{P}_L = \frac{q^2\dot{\beta}^2}{16\pi^2\varepsilon_0 c}\int d\phi \int d\theta \sin^2\theta \sin\theta$$

Which, with a substitution[3], gives:

$$\bar{P}_L = \frac{q^2\dot{\beta}^2}{6\pi\varepsilon_0 c}$$

3.3.3 Radiation: Acceleration & Velocity Perpendicular

This corresponds to a charged particle being accelerated in a circular orbit. This will correspond to $\dot{\boldsymbol{\beta}} \cdot \boldsymbol{\beta} = 0$

Let us start with our general formula for radiation:

$$\frac{d\bar{P}}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\left[\frac{\left(\hat{\boldsymbol{R}} \times (\hat{\boldsymbol{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right)^2}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^5}\right]_{ret}$$

Let us, for convenience, write this in the following way:

$$\frac{d\bar{P}}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\left[\frac{\Xi^2}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^5}\right]_{ret} \qquad \Xi \equiv \hat{\boldsymbol{R}} \times (\hat{\boldsymbol{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}$$

Let us expand out the top, using a vector identity as we go:

$$\begin{aligned}\Xi &= \hat{\boldsymbol{R}} \times \hat{\boldsymbol{R}} \times \dot{\boldsymbol{\beta}} - \hat{\boldsymbol{R}} \times \boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} \\ &= (\hat{\boldsymbol{R}}\cdot\dot{\boldsymbol{\beta}})\hat{\boldsymbol{R}} - \dot{\boldsymbol{\beta}} - (\hat{\boldsymbol{R}}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta} + (\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}} \\ &= \hat{\boldsymbol{R}}\cdot\dot{\boldsymbol{\beta}}(\hat{\boldsymbol{R}} - \boldsymbol{\beta}) - \dot{\boldsymbol{\beta}}(1 - \hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})\end{aligned}$$

Hence, as the numerator in the expression for radiation is squared, let us square our expression (the very last one):

$$\Xi^2 = \left(\hat{\boldsymbol{R}}\cdot\dot{\boldsymbol{\beta}}(\hat{\boldsymbol{R}} - \boldsymbol{\beta}) - \dot{\boldsymbol{\beta}}(1 - \hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})\right)^2 = (\hat{\boldsymbol{R}}\cdot\dot{\boldsymbol{\beta}})^2(\hat{\boldsymbol{R}} - \boldsymbol{\beta})^2 + \dot{\beta}^2(1 - \hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})^2 - 2\xi$$

[3] See the appendix for a note on doing such integrals

Where ξ is the 'cross-term' that always results from squaring a bracket:

$$\xi \equiv [(\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})(\hat{\boldsymbol{R}} - \boldsymbol{\beta})] \cdot [\dot{\boldsymbol{\beta}}(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})]$$

Let us compute this, carefully (remembering that terms that are already 'dotted' are a scalar, and are just treated as 'numbers'):

$$\begin{aligned} \xi &\equiv [(\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})(\hat{\boldsymbol{R}} - \boldsymbol{\beta})] \cdot [\dot{\boldsymbol{\beta}}(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})] \\ &= (\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}} - (\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})(\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})(\hat{\boldsymbol{R}} \cdot \boldsymbol{\beta}) \\ &\quad -(\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}} + (\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})(\hat{\boldsymbol{R}} \cdot \boldsymbol{\beta}) \end{aligned}$$

Now, from the setup of the system, we have that velocity and acceleration are perpendicular. Hence, as previously stated $\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}} = 0$. Thus, noting this results in the last two terms above being zero; leaving us with the first two (which we also clean up):

$$\xi = (\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})^2 - (\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})^2(\hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})$$

Taking out a common factor:

$$\xi = (\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})^2(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})$$

Therefore, the squared expression for the numerator is:

$$\begin{aligned} \Xi^2 &= (\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})^2(\hat{\boldsymbol{R}} - \boldsymbol{\beta})^2 + \dot{\beta}^2(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})^2 - 2\xi \\ &= (\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})^2(\hat{\boldsymbol{R}} - \boldsymbol{\beta})^2 + \dot{\beta}^2(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})^2 - 2(\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})^2(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta}) \end{aligned}$$

Again, let us take out a common factor:

$$\begin{aligned} \Xi^2 &= (\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})^2 \left[(\hat{\boldsymbol{R}} - \boldsymbol{\beta})^2 - 2 + 2\hat{\boldsymbol{R}} \cdot \boldsymbol{\beta}\right] + \dot{\beta}^2(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})^2 \\ &= (\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})^2 \left[1 - 2\hat{\boldsymbol{R}} \cdot \boldsymbol{\beta} + \beta^2 - 2 + 2\hat{\boldsymbol{R}} \cdot \boldsymbol{\beta}\right] + \dot{\beta}^2(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})^2 \\ &= -(\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}})^2(1 - \beta^2) + \dot{\beta}^2(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{\beta})^2 \end{aligned}$$

Now, treating $\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}}$ as the projection of $\dot{\boldsymbol{\beta}}$ onto the x-axis, we have that $\hat{\boldsymbol{R}} \cdot \dot{\boldsymbol{\beta}} = \dot{\beta} \sin\theta \cos\phi$, Also, we have that $\hat{\boldsymbol{R}} \cdot \boldsymbol{\beta} = \beta \cos\theta$. Hence, the above becomes:

$$\Xi^2 = -(\dot{\beta} \sin\theta \cos\phi)(1 - \beta^2) + \dot{\beta}^2(1 - \beta\cos\theta)^2$$

Also, we see that the denominator term in the radiation expression can be written:

$$(1 - \boldsymbol{\beta} \cdot \hat{\boldsymbol{R}})^5 = (1 - \beta\cos\theta)^5$$

Hence, using all this, our radiation expression, for acceleration being perpendicular to velocity, is:

$$\begin{aligned} \frac{d\bar{P}}{d\Omega} &= \frac{q^2}{16\pi^2\varepsilon_0 c} \frac{\Xi^2}{(1 - \boldsymbol{\beta} \cdot \hat{\boldsymbol{R}})^5} \\ &= \frac{q^2}{16\pi^2\varepsilon_0 c} \frac{-(\dot{\beta} \sin\theta \cos\phi)(1 - \beta^2) + \dot{\beta}^2(1 - \beta\cos\theta)^2}{(1 - \beta\cos\theta)^5} \end{aligned}$$

Now, to expand out the top, and rearrange:

$$\frac{d\bar{P}}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

Where we have noted that:

$$\gamma^2 = \frac{1}{1-\beta^2}$$

Now, we call this type of radiation *synchrotron radiation*, and we hence have the final expression:

$$\frac{d\bar{P}}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \tag{3.39}$$

3.3.4 Radiation: Summary

We have derived the radiated power, per unit solid angle, for the cases where the velocity and acceleration are parallel and perpendicular.

For the case of parallel, we call the motion *colinear acceleration*, and usually denote the radiation $d\bar{P}_{//}$. The expression we derived was:

$$\frac{d\bar{P}_{//}}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\left[\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^5}\right] \tag{3.40}$$

For the case of perpendicular, we call the radiation *synchrotron radiation*, denote it $dP_\perp$; and the expression we derived was:

$$\frac{d\bar{P}_\perp}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \tag{3.41}$$

3.3.5 Example: Minimum & Maximum Radiation

Let us consider two exercises:

- Let us show where no synchrotron radiation occurs, for $\phi = 0$. Let us also show where maximum radiation occurs.
- Let us find the total radiated power, for both colinear and synchrotron radiation.

No synchrotron radiation Let us set $\phi = 0$ in the expression for $\frac{d\bar{P}_\perp}{d\Omega}$. This gives:

$$\frac{d\bar{P}_\perp}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

If there is to be no radiation, this expression must be set to zero. That is:

$$\frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta}{\gamma^2(1-\beta\cos\theta)^2}\right] = 0$$

We must then solve this for θ. So, expanding out the (top) brackets, using $\sin^2\theta = 1-\cos^2\theta$, and the definition of γ, this becomes:

$$\frac{1}{(1-\beta\cos\theta)^5}\left[(1-\beta\cos\theta)^2-(1-\cos^2\theta)(1-\beta^2)\right]=0$$

Thus, expanding out the top bracket again, and cleaning up, gives:

$$\frac{(\beta-\cos\theta)^2}{(1-\beta\cos\theta)^5}=0$$

Hence, we see that we must have $\cos\theta=\beta$. That is, we have derived that there is no synchrotron radiation, along $\phi=0$, if $\theta_{min}=\cos^{-1}\beta$.

To show where power is a maximum, we must differentiate the $d\bar{P}_\perp$, with respect to θ, then set to zero; and solve for θ. We shall not do this here.

Total power radiated Let us find the total power radiated, first for the colinear case. So, we must integrate the expression over solid angles. Thus:

$$\int\frac{d\bar{P}_{//}}{d\Omega}d\Omega=\frac{q^2}{16\pi^2\varepsilon_0 c}\int\left[\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^5}\right]d\Omega$$

Remebering that $d\Omega=\sin\theta d\theta d\phi$, this becomes:

$$\bar{P}_{//}=\frac{q^2\dot{\beta}^2}{16\pi^2\varepsilon_0 c}\int\int\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\sin\theta d\theta d\phi$$

That is:

$$\bar{P}_{//}=\frac{q^2\dot{\beta}^2}{16\pi^2\varepsilon_0 c}2\pi\int_0^\pi\frac{\sin^3\theta}{(1-\beta\cos\theta)^5}d\theta$$

Making the substitution of $x=\cos\theta$, the integral bit is:

$$\int_{-1}^{1}\frac{1-x^2}{(1-\beta x)^5}dx=\frac{4}{3}\frac{1}{(1-\beta^2)^3}$$

Hence:

$$\bar{P}_{//}=\frac{q^2\dot{\beta}^2}{8\pi\varepsilon_0 c}\frac{4}{3}\frac{1}{(1-\beta^2)^3}$$

And, noting that $1/\gamma^2=(1-\beta^2)$, this is:

$$\bar{P}_{//}=\frac{q^2}{6\pi\varepsilon_0 c}\dot{\beta}^2\gamma^6$$

And, with reference to (3.11); the Larmor total power radiated, this is:

$$\bar{P}_{//}=\bar{P}_L\gamma^6 \qquad \bar{P}_L\equiv\frac{q^2}{6\pi\varepsilon_0 c}\dot{\beta}^2$$

Now, if we do a similar integration, which is more complicated this time, for the synchrotron radiation expression, we find:

$$\bar{P}_\perp=\bar{P}_L\gamma^4$$

3.3.6 Example: Charged Particle in Circular Orbit

Suppose we have a charged particle, in a circular orbit (radius R). We have an applied magnetic field B which keeps the particles in such an orbit. Then, we have that the acceleration is perpendicular to the velocity. Hence, we can use the following expression for the total power radiated:

$$\bar{P}_\perp = \frac{q^2}{6\pi\varepsilon_0 c}\dot{\beta}^2\gamma^4$$

The velocity is $\boldsymbol{v} = c\beta$, and is direction at a tangent to the circle. The acceleration is $\boldsymbol{a} = c\dot{\boldsymbol{\beta}}$, and is directed from the particle to the centre of the orbit. Thus, using this expression:

$$\bar{P}_\perp = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3}\gamma^4$$

Now, we have the relation $a = \frac{v^2}{R}$, from standard Newtonian mechanics for motion in a circle. And also, note that $v = c\beta$. Hence:

$$\bar{P}_\perp = \frac{q^2 c\beta^4}{6\pi\varepsilon_0 R^2}\gamma^4$$

Now, from Newtons force equalling the Lorentz force:

$$\gamma m a = qvB \quad \Rightarrow \quad \gamma m\frac{\beta^2 c^2}{R} = q\beta cB$$

Noting that everything is at right-angles. This easily gives:

$$R = \frac{\gamma m c\beta}{qB}$$

Hence, using this in $\bar{P}_\perp$:

$$\bar{P}_\perp = \frac{q^4\beta^2 B^2}{6\pi\varepsilon_0 m^2 c}\gamma^2$$

Hence, we have derived a formula giving the total power radiated by a charged particle, mass m, in a circular orbit (governed by the externally applied magnetic field B), at speed $c\beta$.

3.4 Discussion

So, in this section, we have considered the relativistic effect of the finite speed of light. We have seen that a consequence of light having a finite speed, is that moving charges radiate. Infact, what we have also seen, is that only accelerated charges radiate. We saw that this was because stationary charges do not radiate, and that we can always put a charge moving with constant velocity into an inertial frame, in which it is at rest.

We then derived the Lienard-Wiechert potentials for point charges in motion, and stated the associated fields.

Finally, we derived a general theorem of radiation, and specialised it to the cases of linear and perpendicular motion; whilst finding positions of maximum and minimum radiation.

This concludes our discussion on retarded potentials; and we proceed onto relativistic electrodynamics.

4 Relativistic Electrodynamics

The aim of this section is to show that the theory of electrodynamics is consistent with the special theory of relativity

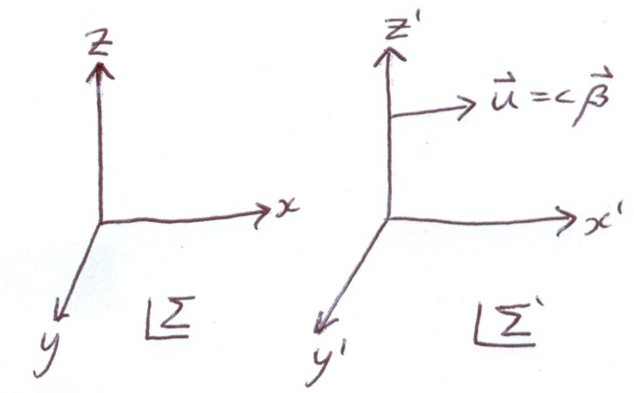

Figure 8: The standard setup for the Lorentz boost along the x-direction. It is standard to state that the origins of the two systems coincide at $t = 0$, and that the motion of the second frame Σ' is inertial relative to the first Σ; where the primed frame is at a speed $\boldsymbol{u} = c\boldsymbol{\beta}$.

Let us state the Galilean transformations:

$$\begin{cases} t = & t' \\ x = & x' + ut' \\ y = & y' \\ z = & z' \end{cases} \tag{4.1}$$

And also the Lorentz transformations:

$$\begin{cases} t = & \gamma(t' + \frac{\beta}{c}x') \\ x = & \gamma(x' + \beta ct') \\ y = & y' \\ z = & z' \end{cases} \tag{4.2}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad \beta \equiv \frac{v}{c} \tag{4.3}$$

These two sets of transformations are for 'boosts along the x-axis'; and are the set we call the "inverse transformation".

We can show that Maxwell's equations are not invariant under Galilean transformations, but are invariant under Lorentz transformations. By 'invariance', we mean that the equations have the same form in both the primed & unprimed frame. We could also show that Newton's equation $\boldsymbol{F} = m\ddot{\boldsymbol{a}}$ is invariant under Galilean, but not under Lorentz transformation.

The postulates of special relativity:

- All laws of nature are independent of the translational motion of the system as a whole;
- Information cannot travel faster than the speed of light.

There are many physical consequences of these postulates, including length contraction, and time dilation.

4.1 Notation

We may be able to see, that from the Lorentz transformation equations, that the following is invariant:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \equiv d^2$$

That is, the modulus of a 4-vector remains unchanged under Lorentz transformation. We shall denote a (contravariant; we shall come to this later) 4-vector by $\vec{x}$, as opposed to the 3-vectors by $\boldsymbol{v}$. So, the above modulus is:

$$\vec{x} \cdot \vec{x} = \vec{x}' \cdot \vec{x}' = d^2$$

Minkowski space 'mixes' the spatial components with a temporal component. Minkowski space is a space in which 4-vectors reside, having their modulus invariant under Lorentz transformation. We shall demonstrate this below.

Let us write the following[4] (what we will call contravariant) 4-vectors:

$$\vec{x} = (x^0, x^1, x^2, x^3) \qquad \vec{y} = (y^0, y^1, y^2, y^3)$$

As we shall also see later, these contravariant vectors occupy the *vector space*. That is, they are what we usually call 'vectors'. Also, by way of notation, we can refer to the vector by $\vec{x}$, or by the set of components $\{x^\mu\}$. We will usually be sloppy with using this notation, so that writing x^μ implies the whole vector. However, in calculations, we will usually use x^μ to refer to that single component. By writing x^μ we are not appealing to any basis system, such as Cartesian or spherical polar.
The scalar product between the two vectors is then written:

$$\vec{x} * \vec{y} = \sum_{\nu,\mu=0}^{3} x^\mu g_{\mu\nu} y^\nu \tag{4.4}$$

Where we have introduced the *metric tensor of Minkowski space*:

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.5}$$

It is important to note that the expression $g_{\mu\nu}$ refers to the element, not the tensor itself, which is why we used $[g_{\mu\nu}]$ above to refer to the whole tensor. Again, we will be sloppy with notation.

[4] I have made a fairly comprehensive treatise of index notation & the various types of vectors, as well as preliminary tensor calculus notes. They may be found on *http://myweb.tiscali.co.uk/jonathanp*. The most useful document is "*Index Notation*", which does use a different metric signature of $(-,+,+,+)$, rather than the $(+,-,-,-)$ used here.

This may also be written as $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Under the Einstein summation convention of implied summation over the available coordinate space, the scalar product may then be written:

$$\vec{x} * \vec{y} = x^\mu g_{\mu\nu} y^\nu$$

Or, the modulus of a 4-vector:

$$d^2 \equiv \vec{x} * \vec{x} = x^\mu g_{\mu\nu} x^\nu$$

This way of writing a modulus (or inner product), in terms of a metric, is very useful, as it has removed the need of how the basis vectors 'interact' with each other. Consider the following:
In Cartesian space, we have the metric δ_{ij}, which is the identity matrix. We shall write the dot product between two vectors in Cartesian space:

$$\boldsymbol{x} \cdot \boldsymbol{y} = (x_i \boldsymbol{e}_i) \cdot (y_j \boldsymbol{e}_j) = x_i y_j \boldsymbol{e}_i \cdot \boldsymbol{e}_j$$

Now, we cant go any further without stating how the basis vectors 'interact'. The Cartesian basis vector have the following relation:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

Then, we may write a Cartesian dot-product, in terms of this rule:

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_i y_j \delta_{ij} = x_i y_i$$

Which is the familiar rule for finding the dot-product of a vector. Now, the first thing to note is that we have used lower indices everywhere, and that they are all latin letter (as opposed to greek). This convention will be used throughout. Second to note, is that instead of writing the vectors as a sum of components with basis (i.e. $\boldsymbol{x} = x_i \boldsymbol{e}_i$), we could ignore the basis. In this way, the dot-product is written simply as:

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_i y_j \delta_{ij}$$

With no mention as to the nature of the basis vectors. Infact, this is technically incorrect: the fact that we have used the expression δ_{ij} means that we have used a pre-conceived rule for the basis vectors. Generally, the rule is called a 'metric', and is usually written g_{ij}. So, the metric for Cartesian space is $g_{ij} = \delta_{ij}$. Then, the dot-product is:

$$\boldsymbol{x} \cdot \boldsymbol{y} = g_{ij} x_i y_j$$

Then, in writing it like this, we have seemingly generalised to any space, without need for a basis, or how the basis interact (information of which is actually 'inside' the metric). So, in Minkowski space (as above), the dot-product of two 4-vectors is written in terms of just the components (i.e. not their basis; we have not considered once what the basis is) and the metric:

$$\vec{x} \cdot \vec{y} = g_{\mu\nu} x^\mu y_\nu$$

Where we have gone back to worrying about upper and lower indices. Let us now consider what these correspond to.

Now, let us define a *contravariant* position vector thus:

$$\vec{x} = x^\mu = (ct, x, y, z) \tag{4.6}$$

And let us define a *covariant* position vector:

$$\vec{x}^d = x_\mu = (ct, -x, -y, -z) \tag{4.7}$$

These two objects may appear to be the same, but they are not! What we used to call just a 'vector', is actually a contravariant vector. These types of vectors transform in a very different way. Let us make the following identification:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{4.8}$$
$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z) = (x^0, -x^1, -x^2, -x^3) \tag{4.9}$$

Now, its worth noting how the metric acts upon co- and contravariant vectors. Let us multiply the metric by a contravariant vector:

$$g_{\mu\nu}x^\nu \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

Now, notice that the expression on the far right is actually how we defined a covariant vector in (4.9). Hence:

$$x_\mu = g_{\mu\nu}x^\nu \tag{4.10}$$

Being careful with the order of the indices[5]. Note that the operation of the metric *lowers* the index, then relabels it. Similarly, we have:

$$x^\mu = g^{\mu\nu}x_\nu \tag{4.11}$$

That is, *raising* the index, by acting upon a covariant vector with the inverse metric (which we will come to shortly).

Now, let us refer back to the expression we had for the modulus of a 4-vector:

$$\vec{x} * \vec{x} = x^\mu g_{\mu\nu} x^\nu$$

Now, we have an expression for 'what happens' if we multiply a contravariant vector by the metric, in (4.10). Hence, let us use that:

$$x^\mu g_{\mu\nu} x^\nu = x^\mu x_\mu$$

We denote this, in terms of the actual vectors (as we used the '$*$' sign for the scalar product of two contravariant vectors) as:

$$\vec{x} \cdot \vec{x}^d = x^\mu x_\mu$$

And this is the equivalent of the 'dot-product we knew before'. This is what we will call "the interval". So, for later reference, for infinitesimals:

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$$

[5] Again, for explanations of these indices, and ways of manipulating them, I refer you to the afore mentioned "*Index Notation*".

Consider a "normal" 3-vector $\boldsymbol{x}$. Its length is given by the dot-product with itself $x^2 = \boldsymbol{x} \cdot \boldsymbol{x}$. Then, given a vector of infinitesimals in each direction:

$$d\boldsymbol{x} = (dx, dy, dz)$$

Its length is obviously:

$$ds^2 \equiv d\boldsymbol{x} \cdot d\boldsymbol{x} = dx^2 + dy^2 + dz^2$$

Hence we see the reasoning behind calling the quantity $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ the 'interval'. We shall come back to this later, upon discussion of proper time.

The inverse of the metric $g_{\mu\nu}$ is written $g^{\mu\nu}$. The inverse metric tensor has the same elements as the non-inverse metric, but it is not the same tensor. So, we may write that $g_{\mu\nu} = g^{\mu\nu}$, but that merely means that its elements are the same, not that the tensors are identical. This highlights the difference between representing a tensor by a matrix, and the tensor itself: we represent a tensor, but we do not write the tensor itself. Also notice that $g_{\mu\nu} = g_{\nu\mu}$: the tensors are symmetric.
We can arrive at an interesting relation by multiplying (4.10) by something that will raise the index on the LHS:

$$g^{\rho\mu} x_\mu = g^{\rho\mu} g_{\mu\nu} x^\nu$$

Now, we can evaluate the LHS, via (4.11), to give:

$$x^\rho = g^{\rho\mu} g_{\mu\nu} x^\nu$$

Now, we must make this consistently true, which leads us to the relation:

$$g^{\rho\mu} g_{\mu\nu} = \delta^\rho_\nu \tag{4.12}$$

Which, if we plug it back in, results in:

$$x^\rho = \delta^\rho_\nu x^\nu = x^\rho$$

Which is indeed true! Notice, we can think about this in a slightly different, but more 'hand-wavey' way: a matrix multiplied by its inverse gives the identity matrix; which is indeed the statement of (4.12).

We say that multiplication always occurs between a covariant and contravariant vector (or vice-versa). We form the following scalar product between two 4-vectors:

$$\vec{x} \cdot \vec{y}^d = x^\mu y_\mu = g^{\mu\rho} x_\rho g_{\mu\nu} y^\nu = g^{\mu\rho} g_{\mu\nu} x_\rho y^\nu = \delta^\rho_\nu x_\rho y^\nu = x_\nu y^\nu$$

Where we have just showed that $x^\mu y_\mu = x_\mu y^\mu$. To do so, we raised/lowered the indices, depending on their initial location. Then, notice the inverse-non multiplication, resulting in the Kronecker-delta, which we then evaluated.
By way of getting 'used' to playing around with these objects, consider the following expression:

$$g_{\mu\rho} g^{\rho\lambda} g_{\lambda\nu}$$

We notice that the middle and last term are inverse times non-inverse, giving:

$$g_{\mu\rho} g^{\rho\lambda} g_{\lambda\nu} = g_{\mu\rho} \delta^\rho_\nu$$

And we can then use the Kronecker-delta:

$$g_{\mu\rho}\delta^{\rho}_{\nu} = g_{\mu\nu}$$

Hence, we have shown:

$$g_{\mu\rho}g^{\rho\lambda}g_{\lambda\nu} = g_{\mu\nu}$$

We have introduced the concept of two different types of vectors: covariant and contravariant vectors. These are infact two different types of vectors, each occupying their own vector space. When one has a standard vector space, one can always define a dual vector space. An example we may be familiar with, is in quantum mechanics: if we have a state in 'ket-space', so that $|k\rangle$ represents such a vector. We also know that taking the Hermitian conjugate of such a vector gives the 'bra-state' vector $\langle k|$. So here, the vector space is the ket-space, and the dual the bra-space. The act of going from the ket-space to the bra-space, via taking the Hermitian conjugate, is akin to multiplying a contravariant vector by the metric, to get to the covariant space.

So, we have *contravariant vectors* occupying the *vector space*, and *covariant vectors* occupying the *dual vector space*. What always used to call vectors, are occupying the vector space, and are thus contravariant vectors.

As we have seen, the operation of a metric upon a contravariant vector gives covariant vector. So, we can think of the metric as being a way of transferring between the vector space & the dual space (and obviously the other way round via the inverse).
And, by way of (brief) introduction to the next section, a Lorentz transformation keeps within one vector space, and gives the vector referred to a different coordinate system. That is, given a covariant vector, a Lorentz transformation of the covariant vector retains the status of 'covariant', but will give the components of the vector with respect to a different frame of reference within the dual space of covariant vectors.

4.2 Lorentz Transformation

Let us re-state the Lorentz transformations, in a slightly re-jigged way:

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned}$$

Now, consider relabelling the coordinates thus[6]:

$$x^0 \equiv ct \qquad x^1 \equiv x \qquad x^2 \equiv y \qquad x^3 \equiv z$$

[6]It should be obvious that these are not exponents, and are just indices.

Then, we can see that we may write the Lorentz transformations, for a boost along the x-direction, as:

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(-\beta x^0 + x^1) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned}$$

Then, we can see that we can group these equations together into a single set of matrix equations:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{4.13}$$

Multiplication of these matrices reveals that we may indeed represent our system in this way. We write this as:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{4.14}$$

Notice that the transformation is symmetric (that is, the transpose of the transformation matrix is the same as the untransposed version):

$$\Lambda^\mu{}_\nu = \Lambda^\nu{}_\mu$$

Now, the next thing we do is a little more subtle. Consider the x'^0 equation:

$$x'^0 = \gamma(x^0 - \beta x^1)$$

Let us differentiate this with respect to x^0, x^1, x^2, x^3 separately:

$$\frac{\partial x'^0}{\partial x^0} = \gamma = \Lambda^0{}_0 \qquad \frac{\partial x'^0}{\partial x^1} = -\gamma\beta = \Lambda^0{}_1 \qquad \frac{\partial x'^0}{\partial x^2} = \frac{\partial x'^0}{\partial x^3} = 0 = \Lambda^0{}_{2,3}$$

We see that these are the elements of the first row of the transformation matrix. Hence, with a little thought, we see that the following is true:

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \tag{4.15}$$

Putting this into (4.14):

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu \tag{4.16}$$

This is actually the definition of a contravariant tensor, of first rank. That is, any quantity A^μ that transforms via:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu \tag{4.17}$$

Is a contravariant tensor, of first rank.

Let us find the inverse transformation. We shall do this by considering the following expression, the lowering of indices, by the metric, in 'transformed' space:

$$x'_\mu = g_{\mu\nu} x'^\nu$$

Then, the far right term we can express in terms of 'unprimed' coordinates, via (4.14), giving:

$$x'_\mu = g_{\mu\nu} \Lambda^\nu{}_\kappa x^\kappa$$

We may then express the far right 'super-scripted' component in terms of a 'subscripted' expression, from (4.11):

$$x'_\mu = g_{\mu\nu} \Lambda^\nu{}_\kappa x^\kappa = g_{\mu\nu} \Lambda^\nu{}_\kappa g^{\kappa\rho} x_\rho$$

Now, a property of transformation matrices is that (we shall justify this in a later section):

$$g_{\mu\nu} \Lambda^\nu{}_\kappa g^{\kappa\rho} = (\Lambda^{-1})_\mu{}^\rho \tag{4.18}$$

Hence, using this:

$$x'_\mu = (\Lambda^{-1})_\mu{}^\rho x_\rho$$

Where we have that:

$$(\Lambda^{-1})_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu} \tag{4.19}$$

Let us write the transformation and the inverse, next to each other, as we would with matrix multiplication (note: we expect the identity matrix out):

$$\Lambda^\mu{}_\nu (\Lambda^{-1})_\rho{}^\nu = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\rho} = \frac{\partial x'^\mu}{\partial x'^\rho} = \delta^\mu_\rho$$

Which indeed conforms to expectation!

In analogue with the contravariant tensor, we have that any covariant tensor of first rank transforms like:

$$B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu \tag{4.20}$$

4.2.1 Differentiation

Let us write the 4-vector differential covariant differential operator:

$$\begin{aligned} \partial_\mu \equiv \frac{\partial}{\partial x^\mu} &= \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) \\ &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \end{aligned}$$

Also, the contravariant differential vector:

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$$

That is (confusingly), the operator ∂_μ is covariant, and ∂^μ is contravariant. The confusion arises when we consider that the differentials themselves have the opposite sign.

It may help to think of the following: consider the grad-operator in Euclidean space:

$$\nabla = \frac{\partial}{\partial x^i}\boldsymbol{e}_i$$

It is the equivalent of saying that ∇ is a covariant vector, but differentiates in a contravariant way. It transforms as a covariant vector.
Notice then, forming the inner product:

$$\partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

That is, we can write the D'Alembertian:

$$\Box^2 = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

We shall frequently use the notation:

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_i}{\partial x_i}$$

Where it is understood that $i = 1, 2, 3$.

We have:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \qquad \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \tag{4.21}$$

4.2.2 Examples

Let us bring together some notation, and use it to show some useful things.

Show that $x^\mu x_\mu = x'^\mu x'_\mu$ To show this, we will show that the RHS is the same as the LHS. To do this, let us write down the transformations:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad x'_\mu = (\Lambda^{-1})_\mu{}^\rho x_\rho$$

Therefore:

$$\begin{aligned} x'^\mu x'_\mu &= \Lambda^\mu{}_\nu x^\nu (\Lambda^{-1})_\mu{}^\rho x_\rho \\ &= (\Lambda^{-1})_\mu{}^\rho \Lambda^\mu{}_\nu x^\nu x_\rho \\ &= \delta^\rho_\nu x^\nu x_\rho \\ &= x_\nu x^\nu \end{aligned}$$

Hence proven.

Show that $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$ To show this, we start with $x_\mu = g_{\mu\nu}x^\nu$. We then multiply the whole thing by $g^{\rho\mu}$:

$$g^{\rho\mu}x_\mu = g^{\rho\mu}g_{\mu\nu}x^\nu$$

Then, raise the indices on the LHS:

$$x^\rho = g^{\rho\mu}g_{\mu\nu}x^\nu$$

Therefore:

$$g^{\rho\mu}g_{\mu\nu} = \delta^\rho_\nu$$

Hence proven.

Show that ∂_μ is a covariant vector Now, this is a little more tricky. We first must write the definitions of ∂_μ and of a covariant vector:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad B'_\nu = \frac{\partial x^\mu}{\partial x'^\nu}B_\mu$$

Now, let $B_\mu \to \partial_\mu$:

$$\frac{\partial x^\mu}{\partial x'^\nu}\partial_\mu = \frac{\partial x^\mu}{\partial x'^\nu}\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\nu} = \partial'_\nu$$

Where we have noticed that the same factors cancel off. Thus, we have shown that:

$$\partial'_\nu = \frac{\partial x^\mu}{\partial x'^\nu}\partial_\mu$$

Hence proven.

4.2.3 Tensors

Let us briefly extend the definition of transformation of tensors.

Consider a contravariant vector. It transforms thus:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}A^\nu$$

This is a rank-1 contravariant tensor. We can motivate a second rank tensor by considering two first rank contravariant tensors:

$$C'^\mu = \frac{\partial x'^\mu}{\partial x^\rho}C^\rho \qquad D'^\nu = \frac{\partial x'^\nu}{\partial x^\lambda}D^\lambda$$

Then, let us write them next to each other:

$$C'^\mu D'^\nu = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\lambda}C^\rho D^\lambda$$

Then, let us define $A^{\mu\nu} \equiv C^\mu D^\nu$; then the above easily becomes the definition of how rank-2 contravariant tensor transforms:

$$A'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\lambda}A^{\rho\lambda} \tag{4.22}$$

To show that a set of quantities $T^{\mu\nu}$ form a contravariant tensor, it is necessary to show that the above holds. It is fairly easy to see the generalisation of this transformation rule to a contravariant tensor of rank-n:

$$A'^{\mu_1\mu_2\mu_3\ldots\mu_n} = \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}}\frac{\partial x'^{\mu_2}}{\partial x^{\nu_2}}\frac{\partial x'^{\mu_3}}{\partial \nu^{\nu_3}}\cdots\frac{\partial x'^{\mu_n}}{\partial x^{\nu_n}}A^{\nu_1\nu_2\nu_3\ldots\nu_n}$$

Similarly, a covariant tensor of rank-n transforms:

$$B'_{\mu_1\mu_2\mu_3\ldots\mu_n} = \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}}\frac{\partial x^{\nu_2}}{\partial x'^{\mu_2}}\frac{\partial \nu^{\nu_3}}{\partial x'^{\mu_3}}\cdots\frac{\partial x^{\nu_n}}{\partial x'^{\mu_n}}B_{\nu_1\nu_2\nu_3\ldots\nu_n}$$

We can also talk of 'mixed rank' tensors. So, for example, a rank-2 contravariant rank-1 covariant tensor would transform like:

$$T'^{\mu\nu}{}_{\rho} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}\frac{\partial x^{\gamma}}{\partial x'^{\rho}}T^{\alpha\beta}{}_{\gamma}$$

Just as we had the metric acting upon a vector, let us show what happens if the metric acts upon a contravariant tensor, of second rank:

$$g_{\mu\rho}A^{\nu\rho} = A^{\nu}{}_{\mu}$$

It drops the index, and relabels it. Notice that we have retained the order of the indices. For a more complicated mixed-rank tensor:

$$g^{\mu\nu}A^{\rho\gamma}{}_{\nu\lambda} = A^{\rho\gamma\mu}{}_{\lambda} \qquad g_{\mu\nu}A^{\rho\nu\lambda}{}_{\kappa} = A^{\rho}{}_{\mu}{}^{\lambda}{}_{\kappa}$$

It is useful to notice:

$$\frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} \qquad \frac{\partial x'^{\mu}}{\partial x'^{\nu}} = \delta^{\mu}_{\nu}$$

And also, with a small amount of thought:

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x^{\lambda}}{\partial x'^{\mu}} = \delta^{\lambda}_{\rho}$$

Although not a precise reason, the above is easily realised if we notice we can cancel out the ∂'^{μ} factors from the top left & bottom right, leaving something we just saw was a Kronecker-delta.

So, a tensor is a set of quantities which transform under a given rule. That is, given one set of quantities, if we know the transformation to some other frame of reference (such as our Lorentz transformation; which is only one example of many), we know what the set of quantities look like, relative to that new frame of reference.

4.2.4 Inverse Lorentz Transformation

Let us consider an alternative way to think about & derive the inverse Lorentz transformation.

We saw that the metric acting on a covariant vector gave a contravariant vector:

$$g^{\mu\nu}x_{\nu} = x^{\mu}$$

We may also operate the metric upon a rank-2 covariant tensor:

$$g^{\mu\nu}A_{\nu\lambda} = A^{\mu}{}_{\lambda}$$

Equivalently, we may act the metric upon a mixed-rank tensor:

$$g^{\mu\nu}A^{\lambda}{}_{\nu} = A^{\lambda\mu}$$

Let us then create a composition:

$$g_{\nu\lambda}g^{\mu\rho}A^{\lambda}{}_{\rho} = g_{\nu\lambda}A^{\lambda\mu} = A_{\nu}{}^{\mu}$$

So, we first acted the inner metric (this order is arbitrary), which raised the ρ, relabelling it μ (notice that the relative order & column is preserved). The outer metric then dropped the λ, relabelling to ν. So, as one operation:

$$A_{\nu}{}^{\mu} = g_{\nu\lambda}g^{\mu\rho}A^{\lambda}{}_{\rho}$$

Now, let us consider that the tensor in question is the Lorentz transformation:

$$\Lambda_{\nu}{}^{\mu} = g_{\nu\lambda}g^{\mu\rho}\Lambda^{\lambda}{}_{\rho}$$

Now, this, as it stands, is not the form in which we have matrix multiplication. Consider $g^{\mu\rho} = (g^{\rho\mu})^T$ (which is infact a fairly pointless operation, since the metric is symmetric. We do it for completeness). Then, replacing the middle metric above with its transpose:

$$\Lambda_{\nu}{}^{\mu} = g_{\nu\lambda}(g^{\rho\mu})^T\Lambda^{\lambda}{}_{\rho}$$

Then, we may reorder the expression (as we are at liberty to: these are just numbers!):

$$\Lambda_{\nu}{}^{\mu} = g_{\nu\lambda}\Lambda^{\lambda}{}_{\rho}(g^{\rho\mu})^T$$

This is now in the form of matrix multiplication:

$$\Lambda' = g\Lambda g^T$$

The prime is to denote that it is in someway different; not that it is in the primed frame! Let us carry out this matrix multiplication; after noting that the transpose of the metric is the same the untransposed metric:

$$\begin{aligned} g\Lambda g^T &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda' \end{aligned}$$

So, upon comparison of this final matrix (that which we called Λ'), we see that it is exactly the matrix containing the elements of the inverse Lorentz tranformation. Hence, we may denote the inverse Lorentz transformation matrix as $\Lambda_{\nu}{}^{\mu}$. We must be very careful with the positions of the indices, as we see that $\Lambda^{\mu}{}_{\nu}$ is the Lorentz transform. Thus, we see the justification of (4.18), as promised! So, common notation, which are completely equivalent, for the inverse Lorentz transformation:

$$\Lambda_{\mu}{}^{\nu} = (\Lambda^{-1})_{\mu}{}^{\nu}$$

4.3 Lorentz 4-Vectors

Now, we have already seen a 4-vector, in the contravariant vector $x^\mu = (ct, \boldsymbol{x})$; where this notation implies that $\boldsymbol{x} = (x, y, z)$, the standard 3-vector. We saw that it had the folowing transformation properties:

$$x^\mu x_\mu = x'^\mu x'_\mu \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Infact, any general 4-vector, with components A^μ has the same properties:

$$A^\mu A_\mu = A'^\mu A'_\mu \qquad A'^\mu = \Lambda^\mu{}_\nu A^\nu$$

So that *any* 4-vector (or two 4-vectors) are invariant under Lorentz transformation. This is infact a condition that a set of quantities must fulfill, in order to be called 4-vectors.

Now, let us define the contravariant infinitesimal:

$$dx^\mu = (cdt, d\boldsymbol{x}) = (cdt, dx, dy, dz)$$

And the corresponding covariant infinitesimal:

$$dx_\mu = (cdt, -d\boldsymbol{x})$$

We then have the line element (which we have already discussed):

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^\mu dx_\mu$$

And which we have shown to be invariant (and is only valid in Minkowski space; due to the form of the metric). Let us write it in terms of the components of the corresponding infinitesimal co- and contravariant vectors:

$$\begin{aligned} ds^2 &= dx^\mu dx_\mu \\ &= c^2 dt^2 + (dx)(-dx) + (dy)(-dy) + (dz)(-dz) \\ &= c^2 dt^2 - d\boldsymbol{x}^2 \end{aligned}$$

And therefore:

$$ds = \sqrt{c^2 dt^2 - d\boldsymbol{x}^2} = \sqrt{dx^\mu dx_\mu}$$

4.3.1 Proper Time

Now, let us define the *proper time* τ as the line element divided by the speed of light. That is, in infinitesimal form:

$$d\tau = \frac{ds}{c} \tag{4.23}$$

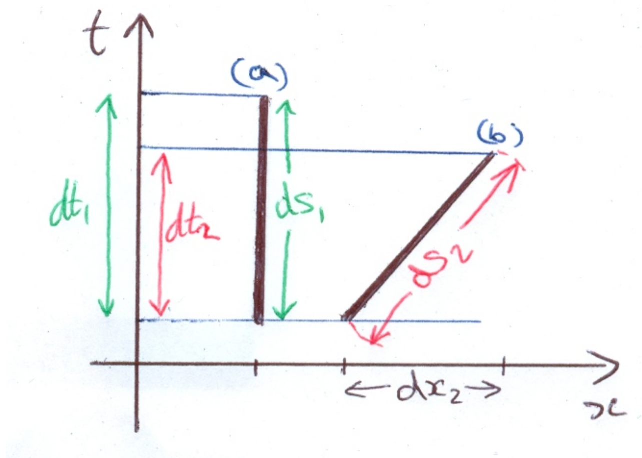

Figure 9: Consider two different situations. Consider (a) first: a particle in its rest frame carves out a vertical line in a space-time diagram: it has motion through time, but not in space. Notice that we have marked on the interval length ds_1 as the length of the line in total, in a space-time diagram; also notice that dt_1 is the amount of time experienced by such a motion. Now consider the second case (b). The particle is now moving through space as well as time. Notice that it has a different projection onto the t axis. A moving particle experiences less time than a stationary particle!

That is:

$$\begin{aligned} d\tau &= \frac{1}{c}\sqrt{dx^\mu dx_\mu} \\ &= \frac{1}{c}\sqrt{c^2dt^2 - d\boldsymbol{x}^2} \\ &= \sqrt{dt^2 - \frac{1}{c^2}d\boldsymbol{x}^2} \\ &= dt\sqrt{1 - \frac{1}{c^2}\left(\frac{d\boldsymbol{x}}{dt}\right)^2} \\ &= dt\sqrt{1-\beta^2} \end{aligned}$$

Where we have noticed that:

$$\left(\frac{d\boldsymbol{x}}{dt}\right)^2 = (\boldsymbol{v})^2 = v \qquad \beta = \frac{v}{c}$$

Also, recall that:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

Hence, the proper time is:

$$d\tau = \frac{dt}{\gamma} \tag{4.24}$$

Let us re-generalise this, by taking ourselves all the way back to the metric-expression:

$$d\tau = \frac{1}{c}\sqrt{g_{\mu\nu}dx^{\nu}dx^{\mu}}$$

Now, we can get a handle on what proper time is, noting the expression $\gamma d\tau = dt$. An observer which is at 'rest' experiences less proper time than an observer 'moving'. That is, proper time not only takes into account movement in space, but also in time. This is the "normal" time-dilation effect, but derived using a metric and spacetime intervals.
So, if we consider the usual scenario of a twin remaining on the earth while another travels on a rocket & back again (ignoring any possible acceleration effects). The proper time experienced by the twin moving through space (i.e. the one on the rocket) is *more* than the twin that did not move through space (i.e. the one on the earth). To think about it another way, consider a ball being thrown, and a ball being rolled along the floor; and that they both have the same starting and ending positions. It is classically obvious that they take different times to traverse the same distance, as the thrown ball must travel up as well as down & across; but the rolled ball only across. So they travel different distances, hence different travel times. It is this exact same thing, except the 'extra distance' travelled is motion through (improper) time.
If you think about sitting stationary; you are carving a line through the 'time axis', but not the spatial axes. The length of the line that joins two events, whether you have moved through space or not, gives the proper time. We can also think about this in terms of possesing a rest-mass energy, even though you are not 'moving'.

The line element ds is the interval of spacetime traversed in proper time $d\tau$.

With reference to Fig (9), we see that if the two lines have the same length (i.e. ds_1 and ds_2 have the same length for both stationary and moving frames), their projections onto the t-axis are different. The stationary line has a longer projection than the moving line. That the lines are the same length is the statement that the proper time for the two observers, is the same.
So, consider the usual analogy of an observer on a rocket ship & another observer stationary on the earth. Consider that both observers age by 5 years. That is, the proper time for both observers is 5 years. However, their projections onto the t-axis is different. The stationary observer has a longer t than the moving observer. Perhaps one can think of it in different terms. Consider that the moving observer starts his journey in the year 2065. The two observers (stationary & moving in the rocket) both agree that the rocket leaves in the year 2065. Now consider that the rocket is able to travel at a very large portion of the speed of light. Then, after the observer in the rocket has aged by (say) 10years, he arrives back to earth. The observer (that was moving) then thinks that the year is 2075. However, the stationary obsever on the earth thinks that the year is 2080. This is the oddity of time dilation!

4.3.2 4-velocity & 4-momentum

Let us consider the 4-velocity: u^{μ}. We shall define it thus:

$$u^{\mu} \equiv \frac{dx^{\mu}}{d\tau} \tag{4.25}$$

Hence, using the fact that $dx^\mu = (cdt, \boldsymbol{x})$:

$$u^\mu = \left(c\frac{dt}{d\tau}, \frac{d\boldsymbol{x}}{d\tau}\right)$$

Now, we see that from (4.24), the first component of the 4-vector is just $u^0 = \gamma c$. Also, consider:

$$\frac{d\boldsymbol{x}}{d\tau} = \frac{d\boldsymbol{x}}{dt}\frac{dt}{d\tau} = \gamma\frac{d\boldsymbol{x}}{dt}$$

Therefore:

$$u^\mu = (c\gamma, \gamma\boldsymbol{u}) \qquad \boldsymbol{u} = \frac{d\boldsymbol{x}}{dt} \tag{4.26}$$

So, let us briefly consider an implication of the 4-velocity vector $u^\mu = (c\gamma, \gamma\boldsymbol{u})$. For an observer who is at rest (spatially), he will posess a component which is progressing along the t-axis at the speed of light. That is, if $\boldsymbol{u} = 0$, then $u^\mu = (c, \boldsymbol{0})$ (after noting that $\gamma = 1$ if $u = 0$). Hence, even a "stationary" particle is moving through the time coordinate.

And, similarly, the 4-momentum is:

$$p^\mu = mu^\mu = (mc\gamma, \gamma m\boldsymbol{u})$$

Now, we also have the relations:

$$E = \gamma mc^2 \qquad \boldsymbol{p} = \gamma m\boldsymbol{u} \qquad E^2 = p^2c^2 + m^2c^4 \tag{4.27}$$

Therefore, we write the 4-momentum as:

$$p^\mu = (E/c, \boldsymbol{p})$$

Now, as p^μ is a 4-vector, we then have that $p^\mu p_\mu = p'^\mu p'_\mu$; an invariant. Now, it is clear that $p_\mu = g_{\mu\nu}p^\nu$, so that:

$$p_\nu = (E/c, -\boldsymbol{p})$$

Therefore:

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 \qquad p^2 \equiv p_x^2 + p_y^2 + p_z^2$$

But, as we said, this is invariant; and so we call it the invariant mass. Hence:

$$\frac{E^2}{c^2} - p^2 = \frac{1}{c^2}(E^2 - c^2p^2) = m^2c^2$$

And therefore, we recover:

$$E^2 - c^2p^2 = m^2c^4$$

So, let us again consider the implication of $p^\mu = (E/c, \boldsymbol{p})$. If a particle is at rest (spatially), then it still posses an inherent energy: its rest energy. The rest energy comes about because the particle still has motion through the time coordinate.

4.3.3 Electrodynamic 4-vectors

Consider the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

It must be valid in all reference frames. Now, let us write the divergence part under the summation convention:

$$\frac{\partial \rho}{\partial t} + \frac{\partial J_i}{\partial x_i} = 0$$

Now, if we consider that we had $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ as our Cartesian position 4-vector, we may think that we can write the above equation in a more compact form. Recall the differential operator:

$$\begin{aligned} \partial_\mu &= \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) \\ &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \end{aligned}$$

Then, if we define the 4-vector current density thus:

$$J^\mu = (c\rho, \boldsymbol{J}) \tag{4.28}$$

Then, the continuity equation becomes:

$$\partial_\mu J^\mu = \frac{1}{c}\frac{\partial c\rho}{\partial t} + \frac{\partial J_i}{\partial x_i} = 0$$

That is, the continuity equation reads:

$$\partial_\mu J^\mu = 0 \tag{4.29}$$

Let us just check that we recover the continuity equation if we write things like this. So:

$$\begin{aligned} \partial_\mu J^\mu &= \frac{\partial J^0}{\partial x^0} + \frac{\partial J_i}{\partial x_i} \\ &= \frac{1}{c}\frac{\partial c\rho}{\partial t} + \frac{\partial J_i}{\partial x_i} \\ &= \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} \\ &= 0 \end{aligned}$$

Which is indeed true.

Recall that the the Lorentz gauge was:

$$\frac{1}{c^2}\frac{\partial \boldsymbol{\phi}}{\partial t} + \nabla \cdot \boldsymbol{A} = 0$$

Then, by analogy with the continuity equation, we see that we can write the 4-vector potential:

$$A^\mu = (\phi/c, \boldsymbol{A}) \tag{4.30}$$

And that the Lorentz gauge can be written:

$$\begin{aligned}\partial_\mu A^\mu &= \frac{\partial A^0}{\partial x^0} + \frac{\partial A_i}{\partial x_i} \\ &= \frac{1}{c}\frac{\partial \phi/c}{\partial t} + \frac{\partial A_i}{\partial x_i} \\ &= \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \frac{\partial A_i}{\partial x_i} = 0\end{aligned}$$

Therefore, we can write the 4-vector potential in a manifestly Lorentez invariant form:

$$\partial_\mu A^\mu = 0 \tag{4.31}$$

Notice, by writing $J^\mu = (c\rho, \boldsymbol{J})$ and $A^\mu = (\phi/c, \boldsymbol{A})$, we have implicitly used the notation that $\boldsymbol{J}$ is the standard 3-vector current density, having components $\boldsymbol{J} = (J_x, J_y, J_z)$; and similiarly for the 3-vector vector-potential $\boldsymbol{A}$. Also, notice that $J^0 = c\rho$, so that even for a charge 'at rest', there exists a charge density. That is, if you look at an electron that appears to be at rest, it is still carving out a line through the 'time dimension', which corresponds to the standard charge density ρ. When it is moving relative to an observer, it then carves out a line in both space and time, and in the process bringing the current density components into play. There is obviously the exact same situation with A^0; a charge at rest still generates a potential field (as we have seen in electrostatics); which we now see as being due to it carving out a line in time. But when it moves, it also produces a vector potential field.

Under the Lorentz gauge, we were able to derive the following wave equations:

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi &= \mu_0 c^2 \rho \\ \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} &= \mu_0 \boldsymbol{J}\end{aligned}$$

Hence, we see that we can write these as (which we then verify):

$$\partial^\nu \partial_\nu A^\mu = \mu_0 J^\mu \tag{4.32}$$

So, that is:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_i^2}\right) A^\mu = \mu_0 J^\mu$$

Let us pick the component $\mu = 0$: $A^0 = \phi/c$ and $J^0 = c\rho$. Then:

$$\begin{aligned}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_i^2}\right) \frac{\phi}{c} &= \mu_0 c\rho \\ \Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_i^2}\right) \phi &= \mu_0 c^2 \rho\end{aligned}$$

Which is indeed the equation corresponding to a wave driven by ρ. Let us pick the component $\mu = j$ (just to be clear, the j^{th} component of the vector potential $\boldsymbol{A}$). Then, $A^j = A_j$ and $J^j = J_j$ (this seems to be odd notation, but it is ok, if one follows the meaning). Then:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_i^2}\right) A^j = \mu_0 c J^j$$

Which is immediately satisfied. Notice that the corresponding wave equation, under summation convention:

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J} \qquad \frac{1}{c^2}\frac{\partial^2 A_j}{\partial t^2} - \frac{\partial^2}{\partial x_i^2} A_j = \mu_0 J_j$$

Let us consider an example:

If ρ_0 is the rest charge density in its rest frame, with $\boldsymbol{J} = 0$, let us find the 4-vector current density in the stationary frame. That is, consider standing on the earth, with a chunk of charge moving, with respect you to as a stationary observer. That is, we have the 'primed' components, and let us find the 'unprimed' components of the 4-vector current.

So, let us say that in the primed frame we have $J'^\mu = (c\rho_0, \mathbf{0})$. We know the transformation:

$$J^\mu = (\Lambda^{-1})_\nu{}^\mu J'^\nu$$

Now, this will actually be easier to see in matrix notation, as we must compute all the components:

$$\begin{pmatrix} J^0 \\ J^1 \\ J^2 \\ J^3 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} J'^0 \\ J'^1 \\ J'^2 \\ J'^3 \end{pmatrix}$$
$$= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\rho_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \gamma c\rho_0 \\ \gamma\beta c\rho_0 \\ 0 \\ 0 \end{pmatrix}$$

That is:

$$J^0 = \gamma c\rho_0 \qquad J^1 = \gamma c\beta\rho_0$$

Recall that:

$$u^\mu = (c\gamma, \gamma\boldsymbol{u})$$

Then:

$$J^\mu = (\gamma c\rho_0, \gamma c\beta\rho_0, 0, 0) = \rho_0(\gamma c, \gamma u, 0, 0)$$

So, taking $\boldsymbol{u} = (u, 0, 0)$, we see that:

$$J^\mu = \rho_0 u^\mu$$

4.4 Electromagnetic Field Tensor

Recall the following expressions of the electric & magnetic fields, in terms of the vectors & scalar potential:

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

Now, let us write the full equation out for the electric field:

$$(E^1, E^2, E^3) = \left(-\frac{\partial \phi}{\partial x}, -\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial z}\right) - \left(\frac{\partial A^1}{\partial t}, \frac{\partial A^2}{\partial t}, \frac{\partial A^3}{\partial t}\right)$$

Picking out the '1' component, say; and writing $x = x^1, y = x^2, z = x^3$:

$$E^1 = -\frac{\partial \phi}{\partial x^1} - \frac{\partial A^1}{\partial t}$$

Now, for generality, suppose we chose component 'i', rather than '1':

$$E^i = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A^i}{\partial t}$$

Now, recall our previous definitions:

$$A^\mu = (\phi/c, A^1, A^2, A^3) \qquad x^\mu = (ct, x^1, x^2, x^3) \qquad x_\mu = (ct, -x^1, -x^2, -x^3)$$

Then, instead of writing ϕ, let us write cA^0. And, instead of having t, let us have cx^0. Thus:

$$E^i = -c\frac{\partial A^0}{\partial x^i} - c\frac{\partial A^i}{\partial x^0}$$

And, remembering that $x^i = -x_i$, and also that $x^0 = x_0$:

$$\frac{E^i}{c} = \frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial x_0}$$

Multiplying by -1:

$$-\frac{E^i}{c} = \frac{\partial A^i}{\partial x_0} - \frac{\partial A^0}{\partial x_i} \equiv F^{0i}$$

That is, we have defined an element of the *electromagnetic field tensor*. Let us continue with the fields. Let us write out the magnetic field:

$$\boldsymbol{B} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ A^1 & A^2 & A^3 \end{vmatrix}$$

So that, picking the component B^1:

$$B^1 = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \frac{\partial A^2}{\partial x_3} - \frac{\partial A^3}{\partial x_2} \equiv F^{32}$$

Similarly:

$$B^2 = \frac{\partial A^3}{\partial x_1} - \frac{\partial A^1}{\partial x_3} \equiv F^{13}$$

Inspection of the order of the indices will show how we have arrived at the labelling we have. Also notice that it is all consistent. So that:

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{4.33}$$

Also notice that we can immediately see that it is antisymmetric:

$$F^{\mu\nu} = -F^{\nu\mu}$$

So, grouping together the terms we have computed, and those we antipicate:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \tag{4.34}$$

Hence, we have the *electromagnetic field tensor*.
As a little aside on anti-symmetry & tensors: if the anti-symmetry of a tensor is defined as $A^{\mu\nu} = -A^{\nu\mu}$ then the components must conform to this. That is, if we represent it as a matrix, reflecting the elements along the diagonal should flip the sign of all components. Now, suppose the tensor had elements along the diagonal. Then, they would not be affected by the flipping, nor the sign flipping. That is, $A^{\mu\mu} = A^{\mu\mu}$, and then we have some components which do not conform to the definition. Therefore, we see that an anti-symmetric tensor must have only zero diagonal entries. Which is something we see from the field tensor.

Now, we have not yet proved that the field tensor does indeed transforms as a tensor. Let us do so. So, we need to prove that the following holds:

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\lambda}} F^{\rho\lambda}$$

As it is supposed to be a contravariant tensor, of second rank. Now, let us prove this by trying to show that the following is true:

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\lambda}} \left(\partial^{\rho} A^{\lambda} - \partial^{\lambda} A^{\rho}\right) = \partial'^{\mu} A'^{\nu} - \partial'^{\nu} A'^{\mu}$$

So, let us consider:

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\lambda}} \partial^{\rho} A^{\lambda}$$

Now, we know that each of the following holds (using the inverse transformation):

$$\partial^{\rho} = \frac{\partial x^{\rho}}{\partial x'^{\kappa}} \partial'^{\kappa} \qquad A^{\lambda} = \frac{\partial x^{\lambda}}{\partial x'^{\sigma}} A'^{\sigma}$$

Thus, using these:

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\lambda}} \partial^{\rho} A^{\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\lambda}} \frac{\partial x^{\rho}}{\partial x'^{\kappa}} \partial'^{\kappa} \frac{\partial x^{\lambda}}{\partial x'^{\sigma}} A'^{\sigma}$$

Tidying up a little:

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\lambda}} \frac{\partial x^{\rho}}{\partial x'^{\kappa}} \frac{\partial x^{\lambda}}{\partial x'^{\sigma}} \partial'^{\kappa} A'^{\sigma}$$

Now, we proceed by trying to make things into Kronecker-deltas. We can use the rule that $\frac{\partial x^{\rho}}{\partial x^{\rho}} = 1$, which is pretty obvious; but we use it over other differentials. We then notice that the first and third; and second and fourth term can be written like this. Thus, giving:

$$\frac{\partial x'^{\mu}}{\partial x'^{\kappa}} \frac{\partial x'^{\nu}}{\partial x'^{\sigma}} \partial'^{\kappa} A'^{\sigma}$$

However, each of the differentials left over are just Kronecker-deltas:

$$\delta^{\mu}_{\kappa}\delta^{\nu}_{\sigma}\partial'^{\kappa}A'^{\sigma} = \partial'^{\mu}A'^{\nu}$$

And therefore, what we have shown is that:

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\lambda}}\partial^{\rho}A^{\lambda} = \partial'^{\mu}A'^{\nu}$$

Confirming the status of $\partial^{\mu}A^{\nu}$ as a contravariant tensor of second rank. And although not a proof, it is not a stretch of the mind to say that $\partial^{\nu}A^{\mu}$ also transforms as a second rank contravariant tensor. And therefore, $F^{\mu\nu}$ is a contravariant tensor, of second rank:

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\lambda}}F^{\rho\lambda} \tag{4.35}$$

This is, by definition of our Lorentz transforms:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\lambda}F^{\rho\lambda}$$

Infact, this is a semi-trivial proof, if we consider how $F^{\mu\nu}$ is constructed, and the construction of (4.22).

4.4.1 Maxwell's Equations from $F^{\mu\nu}$

Now, let us start by stating the following two equations, and then we shall recover Maxwell's equations from them:

$$\partial_{\mu}F^{\mu\nu} = \mu_0 J^{\nu} \tag{4.36}$$

$$\partial^{\mu}F^{\nu\lambda} + \partial^{\nu}F^{\lambda\mu} + \partial^{\lambda}F^{\mu\nu} = 0 \tag{4.37}$$

Firstly, notice that in the last equation, all indices appear in the same order: μ, ν, λ. The subsequent expressions are then just even permutations of this combination.

Now, let us consider the first equation:

$$\partial_{\mu}F^{\mu\nu} = \mu_0 J^{\nu} \tag{4.38}$$

Now, let us consider the case $\nu = 1$:

$$\partial_{\mu}F^{\mu 1} = \mu_0 J^{1}$$

Then, the LHS expression unpacks slightly to:

$$\sum_{\mu=0}^{3}\partial_{\mu}F^{\mu 1} = \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31}$$

Now, we know all of the elements of this:

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t} \quad \partial_1 = \frac{\partial}{\partial x} \quad \partial_2 = \frac{\partial}{\partial y} \quad \partial_3 = \frac{\partial}{\partial z}$$

And, looking at the field tensor elements:

$$F^{01} = -E_1/c \quad F^{11} = 0 \quad F^{21} = B_3 \quad F^{31} = -B_2$$

And therefore, we have:

$$\sum_{\mu=0}^{3} \partial_\mu F^{\mu 1} = -\frac{1}{c^2}\frac{\partial E_1}{\partial t} + \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}$$

And considering the whole equation together:

$$-\frac{1}{c^2}\frac{\partial E_1}{\partial t} + \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} = \mu_0 J^1$$

Shuffling around a bit:

$$\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} = \mu_0 J^1 + \frac{1}{c^2}\frac{\partial E_1}{\partial t}$$

Now, consider the first component of the following cross-product:

$$(\nabla \times \boldsymbol{B})_x = \left| \begin{matrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{matrix} \right|_x = \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}$$

Also, let us write Maxwell's fourth equation (notice that it is a vector equation, and is actually a set of 3 equations):

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

We see that taking $\nu = 1$ in $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$, we found the first component of the Maxwell equation. It is not too hard to see that $\nu = 1, 2, 3$ will produce the whole set of equations that gives Maxwells fourth equation.

Then, what about $\nu = 0$:

$$\sum_{\mu=0}^{3} \partial_\mu F^{\mu 0} = \mu_0 J^0$$

We see that we will need the elements of the field tensor which are down the first column:

$$F^{00} = 0 \quad F^{10} = E_1/c \quad F^{20} = E_2/c \quad F^{30} = E_3/c$$

Thus (noting that $J^0 = c\rho$):

$$\sum_{\mu=0}^{3} \partial_\mu F^{\mu 0} = \frac{1}{c}\left(\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\right) = \mu_0 c\rho$$

Which is completely equivalent to:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

Therefore, in total, we have seen that taking $\nu = 0$ in $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$ we are able to recover Gauss' law, and taking $\nu = 1, 2, 3$, we recover Amperes law.

Let us consider the second equation which we said represented Maxwells equations:

$$\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0$$

We are trying to extract the following set of four equations:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \cdot \boldsymbol{B} = 0$$

Now, it is slightly easier to see whats going on if we write the following:

$$T^{\mu\nu\lambda} \equiv \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0$$

It is fairly easy to find that T^{023}, T^{013} and T^{012} give the three components to Faradays law; and that T^{123} gives the lack of magnetic monopoles equation. Notice, we could have reduced this equation to $T^{\mu\nu\lambda} = 0$, but for it to make sense, we would have to define it anyway, so we leave it in the above form. It is possible to show that $T^{\mu\nu\lambda}$ transforms as a contravariant tensor of third rank; and is proven in the appendix.

4.5 Lorentz Transformations of the Fields

Now, we have that the electromagnetic field tensor transforms thus:

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda F^{\rho\lambda}$$

If we actually wish to compute the elements of the field tensor, in the primed frame, in terms of components in the unprimed frame, we can either go through many many summations, evaluating the above explicitly, or, compute it via matrices. Notice that the above transformation may be written:

$$F' = \Lambda F \Lambda^T$$

We must write it like this to allow us to use matrix multiplication. We can immediately start to do this:

$$F' = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Doing the matrix multiplication of the two far right matrices:

$$F' = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma\beta E_1/c & -\gamma E_1/c & -E_2/c & -E_3/c \\ \gamma E_1/c & -\gamma\beta E_1/c & -B_3 & B_2 \\ \gamma E_2/c - \gamma\beta B_3 & -\gamma\beta E_2/c + \gamma B_3 & 0 & -B_1 \\ \gamma E_3/c + \gamma\beta B_2 & -\gamma\beta E_3/c - \gamma B_2 & B_1 & 0 \end{pmatrix}$$

And the final multiplications:

$$F' = \begin{pmatrix} 0 & -E_1/c & -\gamma(E_2/c - \beta B_3) & -\gamma(E_3/c + \beta B_2) \\ E_1/c & 0 & -\gamma(B_3 - \beta E_2/c) & \gamma(B_2 + \beta E_3/c) \\ \gamma(E_2/c - \beta B_3) & \gamma(B_3 - \beta E_2/c) & 0 & -B_1 \\ \gamma(E_3/c + \beta B_2) & -\gamma(B_2 + \beta E_3/c) & B_1 & 0 \end{pmatrix}$$

Now, the final thing to note is that the field tensor in the primed frame has the same components as the unprimed frame, in its frame. That is, we have:

$$F' = \begin{pmatrix} 0 & -E_1'/c & -E_2'/c & -E_3'/c \\ E_1'/c & 0 & -B_3' & B_2' \\ E_2'/c & B_3' & 0 & -B_1' \\ E_3'/c & -B_2' & B_1' & 0 \end{pmatrix}$$

And therefore we can read off the new components, in terms of the old ones:

$$\begin{array}{cc} E_1' = E_1 & B_1' = B_1 \\ E_2' = \gamma(E_2 - \beta c B_3) & B_2' = \gamma(B_2 + \beta E_3/c) \\ E_3' = \gamma(E_3 + \beta c B_2) & B_3' = \gamma(B_3 - \beta E_2/c) \end{array}$$

Notice: the transformation was for a boost along the x-direction, the field having components E_1, B_1 in that direction. These components are unchanged. It is only the components perpendicular to the boost direction which are changed.

Now, let us get this into vector-form. This is done mainly by inspection. We use that $\boldsymbol{E}' = (E_1', E_2', E_3'); \boldsymbol{E} = (E_1, E_2, E_3)$. By inspection:

$$\boldsymbol{E}' = \gamma\boldsymbol{E} + (1-\gamma)E_1\hat{\boldsymbol{x}} + \gamma\beta c(B_2\hat{\boldsymbol{z}} - B_3\hat{\boldsymbol{y}})$$

Which is infact:

$$\boldsymbol{E}' = \gamma\boldsymbol{E} + \frac{1-\gamma}{v^2}(\boldsymbol{v}\cdot\boldsymbol{E})\boldsymbol{v} + \gamma\boldsymbol{v}\times\boldsymbol{B}$$

And, doing the same thing for the magnetic field:

$$\boldsymbol{B}' = \gamma\boldsymbol{B} + \frac{1-\gamma}{v^2}(\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{v} - \frac{\gamma\boldsymbol{v}\times\boldsymbol{E}}{c^2}$$

4.6 Lienard-Wiechert Fields from Lorentz Transformation

Let us consider two frames: Σ & Σ'. At $t = t' = 0$, the two origins coincide. Now, we consider a charge q at rest within Σ'. Then, its fields in its rest frame is just the Coulomb field:

$$\boldsymbol{E}' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^3}\boldsymbol{r}' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^3}(x', y', z') \qquad \boldsymbol{B}' = 0$$

Now, we wish to compute the fields, from the observation point of Σ. Let us just formulate this in a slightly less mathematical manner:

Consider that an observer is sat inside a box, with a charge at rest inside the box, next to him. That is, an observer is in the rest frame of the charge. Then, the electric field the observer "observes" is just the standard Coulomb field, with no magnetic fields. Then consider that another observer is standing on the surface of the earth, watching the box moving, with the stationary charge inside the box. The observer who is moving relative to the charge (i.e. the one on the earth) "observes" a different field (infact, as we shall see, both electric & magentic) to the observer who is stationary relative to the charge (i.e. the one inside the box). The problem at hand is to compute the electric field that the stationary observer observes, due to that moving charge.

We have seen that we have the following transformations of the field components:

$$\begin{array}{rr} E_1' = E_1 & B_1' = B_1 \\ E_2' = \gamma(E_2 - \beta c B_3) & B_2' = \gamma(B_2 + \beta E_3/c) \\ E_3' = \gamma(E_3 + \beta c B_2) & B_3' = \gamma(B_3 - \beta E_2/c) \end{array}$$

So, noting that $\boldsymbol{B}' = (B_1', B_2', B_3') = (0,0,0) = 0$, we are able to solve a little:

$$B_1 = 0 \qquad B_2 = -\frac{\beta}{c}E_3 \qquad B_3 = \frac{\beta}{c}E_2$$

So, using these:

$$E_1 = E_1' \qquad E_2 = \gamma E_2' \qquad E_3 = \gamma E_3'$$

And therefore:

$$\boldsymbol{E} = (E_1, E_2, E_3) = (E_1', \gamma E_2', \gamma E_3')$$

And, reference to the above Coulomb field, for the charge in its rest frame:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^3}(x', \gamma y', \gamma z')$$

Now we wish to express the components of $\boldsymbol{E}$, in terms of coordinates in Σ. The Lorentz transformations, for the coordinates, at $t = 0$, are just:

$$x' = \gamma(x - vt) = \gamma x \qquad y' = y \qquad z' = z$$

And therefore:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\gamma}{r'^3}(x, y, z)$$

Now, let us finally express the r' in terms of unprimed coordinates. It is evident that:

$$r'^2 = x'^2 + y'^2 + z'^2$$

And that is, by our Lorentz transformations:

$$r'^2 = \gamma^2 x^2 + y^2 + z^2$$

If we use the standard cartesian-spherical polars coordinates transformation, except that we measure axial angles from the x-direction:

$$x = r\cos\theta \qquad y = r\sin\theta\cos\phi \qquad z = r\sin\theta\sin\phi$$

Hence:

$$r'^2 = \gamma^2 r^2 \cos^2\theta + r^2\sin^2\theta$$

Let us write this as:

$$\begin{aligned} r'^2 &= \gamma^2 r^2 \cos^2\theta + r^2 \sin^2\theta \\ &= \gamma^2 r^2 \left(\cos^2\theta + \frac{1}{\gamma^2}\sin^2\theta\right) \\ &= \gamma^2 r^2 \left(\cos^2\theta + (1-\beta^2)\sin^2\theta\right) \\ &= \gamma^2 r^2 (1 - \beta^2 \sin^2\theta) \end{aligned}$$

And therefore, using this:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\gamma}{\gamma^3 r^3(1-\beta^2\sin^2\theta)^{3/2}}\boldsymbol{r}$$

Which is:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q(1-\beta^2)}{r^3(1-\beta^2\sin^2\theta)^{3/2}}\boldsymbol{r}$$

This is the same as the Lienard-Wiechert formula we derived (with considerable vector algebra) for the electric field of a moving point charge, with no acceleration.

4.7 Summary of 4-Vectors & Transformations

So, let us bring together in one place, with little explanation, most of the transformations & metrics discussed.

- Contravariant vector - occupies the vector space:
$$x^\mu = (ct, x, y, z)$$
- Covariant vector - occupies the dual vector space:
$$x_\mu = (ct, -x, -y, -z)$$
- Metric actions - transfer between vector & dual space:
$$x^\mu = g^{\mu\nu} x_\nu \qquad x_\mu = g_{\mu\nu} x^\nu \qquad g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$$
- Lorentz transformation & inverse:
$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \qquad (\Lambda^{-1})_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu} \qquad \Lambda^\rho{}_\mu (\Lambda^{-1})_\nu{}^\mu = \delta^\rho_\nu$$
- Contravariant Lorentzian boost - transform between inertial frames:
$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad x^\mu = (\Lambda^{-1})_\nu{}^\mu x'^\nu$$
- Covariant Lorentzian boost:
$$x'_\mu = (\Lambda^{-1})_\mu{}^\nu x_\nu \qquad x_\mu = \Lambda^\nu{}_\mu x'_\nu$$
- Contravariant vector defintion:
$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$$
- Covariant vector definition:
$$B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu$$
- Contravariant & covariant differential operators:
$$\partial^\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla \right) \qquad \partial_\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla \right)$$
- 4-current & 4-potential:
$$J^\mu = (c\rho, \boldsymbol{J}) \qquad A^\mu = (\phi/c, \boldsymbol{A})$$
- Lorentz gauge, continuity equation & wave equation:
$$\partial_\mu A^\mu = 0 \qquad \partial_\mu J^\mu = 0 \qquad \partial_\mu \partial^\mu A^\nu = \mu_0 J^\nu$$
- Electromagnetic field tensor:
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$
- Maxwell's equations:
$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \qquad \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0$$

4.8 Discussion

In this section, we have formulated electrodynamics in a way which is invariant under Lorentz transformation.

We have introduced index notation, and contra- and co-variant vectors. We have looked at how to transform between the two vector spaces via the Minkowski metric, and how to perform a Lorentz boost, within one space.

We then looked at various 4-vectors, and confirmed that they did transform in an expected way. We then formulated various previously known equations (such as the Lorentz gauge & the continuity equation) in a way which was Lorentz invariant (that is, had the same form in different inertial frames of reference), which is in concordance with one of the postulates of special relativity.

We were then able to define the electromagnetic field tensor, showing that it is indeed a tensor; and find a way of expressing Maxwell's four equations in a very elegant way.

Finally, we found the transformations of the field components; finishing with an example which recovered the Lienard-Wiechert fields that we derived in the previous section; but with considerably less algebra!

A Vector Identities & Tricks

A.1 Vector Identities

Below are some commonly used vector identities:

$$\begin{aligned}
\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) &= \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) \\
\boldsymbol{a}\times\boldsymbol{b}\times\boldsymbol{c} &= (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} \\
\nabla\times\nabla\psi &= 0 \\
\nabla\cdot(\nabla\times\boldsymbol{a}) &= 0 \\
\nabla\times\nabla\times\boldsymbol{a} &= \nabla(\nabla\cdot\boldsymbol{a})-\nabla^2\boldsymbol{a} \\
\nabla\cdot(\psi\boldsymbol{a}) &= \boldsymbol{a}\cdot\nabla\psi+\psi\nabla\cdot\boldsymbol{a} \\
\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a})-\boldsymbol{a}\cdot(\nabla\times\boldsymbol{b}) \\
\nabla\times\boldsymbol{a}\times\boldsymbol{b} &= \boldsymbol{a}(\nabla\cdot\boldsymbol{b})-\boldsymbol{b}(\nabla\cdot\boldsymbol{a})+(\boldsymbol{b}\cdot\nabla)\boldsymbol{a}-(\boldsymbol{a}\cdot\nabla)\boldsymbol{b}
\end{aligned}$$

All are easily proven, but some (if not all) are pretty tedious to actually show!

We have also made use of:

$$\begin{aligned}
\nabla\cdot\frac{\hat{\boldsymbol{r}}}{r^2} &= 4\pi\delta(r) \\
\nabla^2\frac{1}{r} &= -4\pi\delta(r)
\end{aligned}$$

A.1.1 Curl in Spherical & Cylindrical Polars

Suppose we have $\boldsymbol{V}(r,\theta,\phi)$, so that its components are V_r, V_θ, V_ϕ, then, its curl, in spherical polars, is given by:

$$\nabla\times\boldsymbol{V} = \begin{vmatrix} \boldsymbol{e}_r/r^2\sin\theta & \boldsymbol{e}_\theta/r\sin\theta & \boldsymbol{e}_\phi/r \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ V_r & rV_\theta & r\sin\theta V_\phi \end{vmatrix} \tag{A.1}$$

Similarly, in cylindrical polars:

$$\nabla\times\boldsymbol{V} = \begin{vmatrix} \boldsymbol{e}_r/r & \boldsymbol{e}_\phi & \boldsymbol{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ V_r & rV_\phi & V_z \end{vmatrix} \tag{A.2}$$

A.2 Useful Tricks

Below are some useful ‘tricks’ to bear in mind when evaluating certain expressions.

Suppose we have the integral:

$$\mathcal{I} = \int_0^\pi \cos\theta\sin\theta\, d\theta$$

Then, we can evaluate it via the subsitution:

$$x \equiv \cos\theta$$

Then, the lower limit becomes $x = \cos 0 = 1$, and the upper limit $x = \cos\pi = -1$. We also trivially see that $dx = -\sin\theta d\theta$. Hence, let us rewrite the integral:

$$\mathcal{I} = -\int_{1}^{-1} x dx$$

But, swopping the limits:

$$\mathcal{I} = \int_{-1}^{1} x\, dx$$

Which results in zero. For a more general case, consider:

$$\mathcal{I} = \int_{0}^{\pi} \cos^n\theta \sin\theta\, d\theta$$

Then, using the same substitution:

$$\mathcal{I} = \int_{-1}^{1} x^n\, dx = \begin{cases} 0 & n\, odd \\ \frac{2}{n+1} & n\, even \end{cases}$$

So:

$$\int_{0}^{\pi} \cos^n\theta \sin\theta\, d\theta = \begin{cases} 0 & n\, odd \\ \frac{2}{n+1} & n\, even \end{cases}$$

Although I have not checked, this will probably only work for $n > 0$.

B Worked Examples

B.1 Long Beam of Charge - Fields & Force

Consider a long straight beam of electrons, of radius a, and (uniform) line charge density $e\lambda$, travelling along the z axis with velocity. Let us compute the electric and magnetic fields inside & outside the beam; and then compute the force the beam feels.

Let us start by considering the electric field, for $r \geq a$. So, we write Gauss' law:

$$\int_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{1}{\varepsilon_0} \int_V \rho_V dV$$

Which will find the electric field through a surface S, which encloses charge in a volume V.
Let us initially compute the charge enclosed by the beam, if it has length l. So:

$$\int \rho_V dV = \int \frac{\rho_l}{A} A dl$$

Where we have noted that the volume charge density is just the line charge density over the area. Also, the volume is just the area times length. Thus, we have:

$$Q_{enc} = \int \frac{\rho_l}{A} A dl = \lambda e l$$

Hence, using Gauss' law, where the surface is a cylinder, radius r, where $r > a$, and length l. Hence, Gauss' law results in:

$$E_r 2\pi r l = \frac{1}{\varepsilon_0} \lambda e l$$

That is:

$$E_r = \frac{\lambda e}{2\pi\varepsilon_0 r} \quad r > a$$

For inside the beam, we must re-consider the charge enclosed by the Gaussian surface. We see that it is the total charge, multiplied by the area of the surface, and divided by the total area of the beam. That is:

$$Q_{enc} = \frac{\lambda e l}{\pi a^2} \pi r^2$$

Thus, the electric field inside is given by:

$$E_r = \frac{\lambda e r}{2\pi\varepsilon_0 a^2} \quad r < a$$

Let us now compute the magnetic field, using Amperes law:

$$\oint_\ell \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 \int_S \boldsymbol{j} \cdot d\boldsymbol{S}$$

So, let us compute the magnetic field outside. Consider a circle, radius r, which will enclose a current I. However note that $j = v\rho$. Thus: $\boldsymbol{I} = v\lambda e\hat{\boldsymbol{z}}$. Hence:

$$B_\theta 2\pi r = \mu_0 v \lambda e$$

Hence:

$$B_\theta = \frac{\mu_0 v \lambda e}{2\pi r} \quad r > a$$

Let us compute the magnetic field inside. Let us use the same argument as we did for the charge enclosed:

$$I_{enc} = \mu_0 v \lambda e \frac{\pi r^2}{\pi a^2}$$

Hence, the magnetic field inside is:

$$B_\theta = \frac{\mu_0 v \lambda e r}{2\pi a^2} \quad r < a$$

Now, let us compute the force on the beam. We write the Lorenz force law:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Let us compute the force inside the beam. So, the cross product:

$$\boldsymbol{v} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}} \\ 0 & 0 & v \\ 0 & B_\theta & 0 \end{vmatrix} = -\hat{\boldsymbol{r}} v B_\theta$$

Hence:

$$\begin{aligned} F_r &= \lambda e \left(\frac{\lambda e r}{2\pi \varepsilon_0 a^2} - \frac{\mu_0 v^2 \lambda e r}{2\pi a^2} \right) \\ &= \frac{\lambda^2 e^2 r}{2\pi a^2} \left(\frac{1}{\varepsilon_0} - \mu_0 v^2 \right) \\ &= \frac{\lambda^2 e^2 r}{2\pi a^2 \varepsilon_0} \left(1 - \varepsilon_0 \mu_0 v^2 \right) \end{aligned}$$

Now, we also notice that $c^2 = \frac{1}{\varepsilon_0 \mu_0}$. Hence:

$$F_r = \frac{\lambda^2 e^2 r}{2\pi a^2 \varepsilon_0} \left(1 - \frac{v^2}{c^2} \right)$$

Thus, the total force is just:

$$F_r = \int_0^a F_r' dr'$$

Which just results in:

$$F_r = \frac{\lambda^2 e^2}{4\pi \varepsilon_0} \left(1 - \frac{v^2}{c^2} \right)$$

Now, notice, if $v << c$, then the force is large, and the beam spreads out. If $v \approx c$, then the force approaches zero; hence the beam retains its size. High energy particle accelerators take advantage of this when focussing beams.

B.2 Spherical Shell Charge Distribution: $\sigma(\theta) = \sigma_0 \cos\theta$

A spherical cavity of radius a is located in a linear dielecric medium, of permittivity ε_r. On the surface of the cavity, charge is distributed with a surface density $\sigma(\theta) = \sigma_0 \cos\theta$.
Let us write down the relations between scalar potential, electric & displacement fields, at the surface.
Now, the first thing to note, is that the 'cavity' is more a 'bubble'; that is, the medium inside is the same as that outside. Now, at the surface, the electric displacement field has the relation:

$$(\boldsymbol{D}_2 - \boldsymbol{D}_1) \cdot \hat{\boldsymbol{n}} = \rho_s$$

If 1 is inside, and 2 outside, and the unit normal points from the boundary to region '2': outside.

Figure 10: The direction of the normal vector, for shell of distributed charge. Region (1) is inside, and (2) outside.

So, that is:

$$D_{2,\perp} - D_{1,\perp} = \sigma_0 \cos\theta$$

At $r = a$. Also, as we have that $\boldsymbol{D} = \varepsilon_r \varepsilon_0 \boldsymbol{E}$. Hence:

$$\varepsilon_r \varepsilon_0 E_{2,\perp} - \varepsilon_r \varepsilon_0 E_{1\perp} = \sigma_0 \cos\theta$$

Notice, for arguments sake, that if the two regions had different permittivities, then this reads:

$$\varepsilon_{2,r} \varepsilon_0 E_{2,\perp} - \varepsilon_{1,r} \varepsilon_0 E_{1\perp} = \sigma_0 \cos\theta$$

Going back to our system:

$$E_{2,\perp} - E_{1\perp} = \frac{\sigma_0}{\varepsilon_r \varepsilon_0} \cos\theta$$

Now, we also have the relation that $\boldsymbol{E} = -\nabla V$, where V is the scalar potential. Hence, in the radial direction (which is the direction in which $E_\perp$ is in), we thus have:

$$\frac{\partial V_2}{\partial r} - \frac{\partial V_1}{\partial r} = -\frac{\sigma_0}{\varepsilon_r \varepsilon_0} \cos\theta$$

Have in mind that all the above relations are for $r \to a$.

Now, let us compute the monopole and dipole moments, due to the charge distribution. And let us do so for the case that the medium is vacuum. That is, $\chi_E \to 0$, where $\varepsilon_r = 1 + \chi_E$. So, we have that $\varepsilon_r = 1$.

Now, the charge distribution is given by (the axially symmetric):

$$\rho(\boldsymbol{r'}) = \delta(r' - a)\sigma(\theta) = \delta(\boldsymbol{r'} - a)\sigma_0 \cos\theta'$$

The potential, due to a multipole expansion, is given by:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^{\ell} P_\ell(\cos\gamma)\rho(\boldsymbol{r'})d^3r'$$

Where $\gamma \equiv \theta' - \theta$. That is, the monopole contribution is just the $n = 0$ term, and dipole $n = 1$. So, if we denote this as:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} p_\ell$$

Where:

$$p_\ell \equiv \int r'^{\ell} P_\ell(\cos\gamma)\rho(\boldsymbol{r'})d^3r'$$

Where $\gamma \equiv \theta' - \theta$. Note, the volume element is given by:

$$d^3r' = r'^2 \sin\theta' dr' d\theta' d\phi'$$

Then we see that we can compute each 'pole'-contribution separately.

So, total charge (i.e. the monopole moment) is given by:

$$p_0 \equiv q = \int \rho(\boldsymbol{r'})d^3r'$$

That is (noting that $P_0(\cos\gamma) = 1$):

$$\begin{aligned} p_0 &= \int \rho(\boldsymbol{r'})r'^2 \sin\theta' dr' d\theta' d\phi' \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{\infty} \delta(r' - a)\sigma_0 \cos\theta' r'^2 \sin\theta' dr' d\theta' d\phi' \end{aligned}$$

Now, the dr' integral picks out only the point $r' = a$; due to the δ-function. So:

$$p_0 = \sigma_0 a^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \cos\theta' \sin\theta' d\theta' d\phi'$$

That is:

$$p_0 = \sigma_0 a^2 2\pi \int_{\theta=0}^{\pi} \cos\theta' \sin\theta' d\theta'$$

To do this integral, either we notice immediately that the two functions are orthogonal, and hence the integral is zero; or, we do a substitution:

$$x = \cos\theta \quad \Rightarrow \quad dx = -\sin\theta d\theta$$

Hence, the integral (just the integral, ignoring the constants) is:

$$I = -\int_{1}^{-1} x dx = \int_{-1}^{1} x dx = \tfrac{1}{2}(1 - 1) = 0$$

That is, just zero. Hence, the monopole moment is zero:

$$p_0 = 0$$

Now, let us consider the dipole moment. This is given by the integral:

$$\begin{aligned} p_1 &= \int r' P_1(\cos\gamma)\rho(\boldsymbol{r}')d^3r' \\ &= \int\int\int r'\cos\gamma\delta(r'-a)\sigma_0\cos\theta' r'^2\sin\theta' dr' d\theta' d\phi' \end{aligned}$$

Where we have noted that $P_1(\cos\gamma) = \cos\gamma$. Now, we must be careful in continuing. We must remember that:

$$\cos\gamma = \cos(\theta' - \theta) = \cos\theta\cos\theta' + \sin\theta\sin\theta'$$

Hence, the integral becomes:

$$p_1 = \int\int\int r'(\cos\theta\cos\theta' + \sin\theta\sin\theta')\delta(r'-a)\sigma_0\cos\theta' r'^2\sin\theta' dr' d\theta' d\phi'$$

Let us initially evaluate the r', ϕ' integrals; which just give:

$$p_1 = \sigma_0 a^3 2\pi \int_0^\pi (\cos\theta\cos\theta' + \sin\theta\sin\theta')\cos\theta'\sin\theta' d\theta'$$

Notice, we must be careful about what is primed, and un-primed. Of course, this step could have been done without inserting the expression for $\cos\gamma$ first. So, let us split this integral into its two additive parts:

$$\begin{aligned} (i) &\quad \int \cos\theta\cos\theta'\cos\theta'\sin\theta' d\theta' \\ (ii) &\quad \int \sin\theta\sin\theta'\cos\theta'\sin\theta' d\theta' \end{aligned}$$

So that $p_1 = \sigma_0 a^3 2\pi[(i) + (ii)]$. Looking at (i) first:

$$\int_0^\pi \cos\theta\cos\theta'\cos\theta'\sin\theta' d\theta' = \cos\theta\int_0^\pi \cos^2\theta'\sin\theta' d\theta'$$

Which we can do via substitution $x = \cos\theta'$. Which easily gives:

$$\cos\theta\left[-\int_1^{-1} x^2 dx\right] = \cos\theta\left[\int_{-1}^1 x^2 dx\right]$$

Thus:

$$(i) \to \frac{2}{3}\cos\theta$$

Now, (ii):

$$\sin\theta\int_0^\pi \sin^2\theta'\cos\theta' d\theta'$$

We substitute $x = \sin\theta'$:

$$\rightarrow \sin\theta \left[\int_0^0 x^2 dx\right] = 0$$

Hence, $(ii) \rightarrow 0$.

Hence, the dipole moment is:

$$p_1 = \sigma_0 a^3 2\pi \frac{2}{3}\cos\theta$$

That is:

$$p_1 = \frac{4}{3}\sigma_0 \pi a^3 \cos\theta$$

Hence, the calculated contributions to the potential, due to the multipole expansion (i.e. that due to mono- and di-pole) is:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r^1}p_0 + \frac{1}{r^2}p_1\right)$$

And is hence just:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{4}{3}\sigma_0 a^3 \pi \cos\theta$$

That is:

$$V(\boldsymbol{r}) = \frac{\sigma_0 a^3 \cos\theta}{3\varepsilon_0 r^2}$$

It is useful just to point out that $V(\boldsymbol{r})$ actually means that it is a function of all 3 positions; that is: $V(\boldsymbol{r}) = V(r, \theta, \phi)$. Thus, we have computed the scalar potential, up to the dipole term only.

Now, let us compute the exact analytical form of the scalar potential $V(\boldsymbol{r})$, in the case that $\chi_E \neq 0$. We do so for both the regions inside $r \leq a$ and outside $r \geq a$ the 'cavity'.

Now, we have that the solution to Laplace's equation, in spherical polars, with axial symmetry, is of the form:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

We have the case '1' inside, and '2' outside. Inside, we have that the potential must not diverge; hence, we reject its B-term. Similarly, we reject the outsides A term, as that would make the potentials diverge for $r \rightarrow \infty$. Hence, we have:

$$\begin{aligned} V_1 &= \sum_\ell A_\ell r^\ell P_\ell(\cos\theta) \qquad r \leq a \\ V_2 &= \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad r \geq a \end{aligned}$$

Now, to find the constants, let us initially apply the boundary condition that the potential is continuous at the boundary $r = a$. Hence:

$$\sum_\ell A_\ell a^\ell P_\ell(\cos\theta) = \sum_\ell \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos\theta)$$

Now, to find the coefficients, we apply a Fourier-type procedure. Multiply both sides by a 'different' polynomial, $P_{\ell'}(\cos\theta)$, say, and integrate. So, the LHS:

$$\int_0^\pi \sum_\ell A_\ell a^\ell P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta d\theta$$

Now, due to orthogonality of Legendre polynomials, we know that:

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x) dx = \begin{cases} 0 & \ell \neq \ell' \\ \frac{2}{2\ell+1} & \ell = \ell' \end{cases}$$

Hence:

$$\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta d\theta = \begin{cases} 0 & \ell \neq \ell' \\ \frac{2}{2p+1} & \ell = \ell' \end{cases}$$

From this, we can see that we will just pick out, from the sum over ℓ, the term $\ell = \ell'$:

$$\frac{2}{2\ell' + 1} A_{\ell'} a^{\ell'} = \frac{2}{2\ell' + 1} \frac{B_{\ell'}}{a^{\ell'+1}}$$

That is, reverting back to using ℓ, as it is just a dummy index:

$$A_\ell a^\ell = \frac{B_\ell}{a^{\ell+1}}$$

Hence:

$$B_\ell = A_\ell a^{2\ell+1}$$

Now, another boundary condition we have is that of continuity of $D_\perp$ at the boundary. This, we showed, is just:

$$\left[\frac{\partial V_2}{\partial r} - \frac{\partial V_1}{\partial r}\right]_{r=a} = -\frac{1}{\varepsilon_r \varepsilon_0} \sigma_0 \cos\theta$$

That is:

$$\frac{\partial}{\partial r} \sum_\ell \left[\frac{B_\ell}{a^{\ell+1}} P_\ell(\cos\theta) - A_\ell a^\ell P_\ell(\cos\theta)\right] = -\frac{1}{\varepsilon_r \varepsilon_0} \sigma_0 \cos\theta$$

Giving:

$$\sum_\ell \left[\frac{\ell+1}{a^{\ell+2}} B_\ell P_\ell(\cos\theta) + \ell A_\ell a^{\ell-1} P_\ell(\cos\theta)\right] = \frac{1}{\varepsilon_r \varepsilon_0} \sigma_0 \cos\theta$$

Using our derived relation $B_\ell = A_\ell a^{2\ell+1}$, this results in:

$$\sum_\ell \left[\frac{(\ell+1) a^{2\ell+1}}{a^{\ell+2}} A_\ell P_\ell(\cos\theta) + \ell A_\ell a^{\ell-1} P_\ell(\cos\theta)\right] = \frac{1}{\varepsilon_r \varepsilon_0} \sigma_0 \cos\theta$$

Which easily gives:

$$\sum_{\ell=0}^{\infty} (2\ell+1) A_\ell P_\ell(\cos\theta) a^{\ell-1} = \frac{1}{\varepsilon_r \varepsilon_0} \sigma_0 \cos\theta$$

Now, to find the A_ℓ, we employ the same technique as we did before; namely to multiply by a different polynomial, and integrate. We shall be a bit more formal in the integration however, in that we specify the limits as the period of the polynomials. So, the LHS is:

$$\sum_\ell \int_0^\pi (2\ell+1)A_\ell a^{\ell-1} P_\ell(\cos\theta) P_{p'}(\cos\theta)\sin\theta\, d\theta$$

That is equal to, via orthogonality:

$$\sum_\ell (2p+1)A_\ell a^{\ell-1}\delta_{\ell\ell'}\frac{2}{2\ell+1} = 2A_\ell a^{\ell-1}$$

The RHS is equal to, when multiplied and integrated:

$$\frac{1}{\varepsilon_r\varepsilon_0}\sigma_0\cos\theta \to \frac{1}{\varepsilon_r\varepsilon_0}\sigma_0\int_0^\pi \cos\theta P_{\ell'}(\cos\theta)\sin\theta d\theta$$

Which is equivalent to, putting $x = \cos\theta$:

$$\frac{\sigma_0}{\varepsilon_r\varepsilon_0}\int_{-1}^1 xP_\ell(x)dx$$

Hence, we have:

$$2A_\ell a^{\ell-1} = \frac{\sigma_0}{\varepsilon_r\varepsilon_0}\int_{-1}^1 xP_\ell(x)dx$$

Thus, the coefficients are given by:

$$A_\ell = \frac{\sigma_0}{2a^{\ell-1}\varepsilon_r\varepsilon_0}\int_{-1}^1 xP_\ell(x)dx$$

So, considering a few Legendre polynomials, and the coefficents they will generate:

$$\begin{aligned} P_0(x) &= 1 \quad \Rightarrow \quad A_0 = 0 \\ P_1(x) &= x \quad \Rightarrow \quad A_1 = \frac{\sigma_0}{3\varepsilon_r\varepsilon_0} \end{aligned}$$

Infact, all higher A_i are zero (a fact we gloss over here, but will spend more time on in other examples). Thus, the potential inside:

$$V_1 = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta) \quad r \le a$$

Thus:

$$V_1 = \frac{\sigma_0}{3\varepsilon_r\varepsilon_0} r\cos\theta$$

And, outside:

$$V_2 = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \quad r \ge a$$

Using our derived relation $B_\ell = A_\ell a^{2\ell+1}$, this is:

$$V_2 = \sum_\ell \frac{A_\ell a^{2\ell+1}}{r^{\ell+1}} P_\ell(\cos\theta) \quad r \geq a$$

Thus:

$$V_2 = \frac{\sigma_0}{3\varepsilon_r\varepsilon_0}\frac{a^3}{r^2}\cos\theta \quad r \geq a$$

Hence, in summary, we have that the potential in all space is given by:

$$V(r,\theta) = \begin{cases} \frac{\sigma_0}{3\varepsilon_r\varepsilon_0} r\cos\theta & r \leq a \\ \frac{\sigma_0}{3\varepsilon_r\varepsilon_0}\frac{a^3}{r^2}\cos\theta & r \geq a \end{cases}$$

Note, the electric field, inside the cavity is actually constant:

$$E_r = -\frac{\partial V_1}{\partial r} = -\frac{\sigma_0}{3\varepsilon_r\varepsilon_0}\cos\theta$$

That is, the field inside the cavity is uniform.

An alternative way to do this, which is actually a bit easier to see, is to express the charge distribution as a Legendre polynomial. Notice that $\sigma(\theta) = \sigma_0 \cos\theta = \sigma_0 P_1(\cos\theta)$. Then, at the stage of 'finding coefficients', one can immediately see that only the $\ell = 1$ term will contribute, by orthogonality of the P_ℓ's. We demonstrate this method in the next example.

B.2.1 Spherical Shell Charge Distribution: $\sigma(\theta) = \sigma_0 \cos 2\theta$

Suppose we have a spherical-shell charge distribution; so that we have a charge density $\sigma(\theta) = \sigma_0 \cos 2\theta$ at a radius a, in a medium with $\chi_E = 0$. Let us compute the potential, in all space, for such a system.

Let us use a trig identity, to rewrite the charge distribution:

$$\cos 2\theta = 2\cos^2\theta - 1$$

Thus:

$$\sigma(\theta) = \sigma_0(2\cos^2\theta - 1)$$

Now, let us examine this. We note that we can conceive that this is made up of two Legendre polynomials:

$$P_0(\cos\theta) = 1 \qquad P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1)$$

So, to express our charge distribution in terms of the above polynomials, let us suppose:

$$\sigma(\theta) = \sigma_0[\alpha P_0 + \beta P_2]$$

Then, equating:

$$\alpha + \tfrac{\beta}{2}(3\cos^2\theta - 1) = 2\cos^2\theta - 1$$

And, upon examination of the coefficients, we see that:

$$\alpha = -\tfrac{1}{3} \qquad \beta = \tfrac{4}{3}$$

Hence, we have an expression for our charge distribution, in terms of Legendre polynomials:

$$\sigma(\theta) = \frac{\sigma_0}{3}[4P_2(\cos\theta) - P_0(\cos\theta)]$$

This is a useful way to represent the charge distribution, as it will allow us to evaluate the potential expansion exactly. Now, let us write the expansion of the potential:

$$V(r,\theta) = \sum_\ell \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right] P_\ell(\cos\theta)$$

If we want the field to converge (or at least, no diverge) as $r \to 0, \infty$, we must discard the relevant terms, when evaluating the potential inside and outside the sphere. Thus:

$$\begin{aligned} V_1 &= \sum_\ell A_\ell r^\ell P_\ell(\cos\theta) \qquad r \leq a \\ V_2 &= \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad r \geq a \end{aligned}$$

Now, we have that the potential is continuous at $r = a$. Hence:

$$V_1(r = a) = V_2(r = a) \quad \Rightarrow \quad B_\ell = a^{2\ell+1} A_\ell$$

Where we have used a similar argument as we used in the other problem. Now, we have that the electric field is discontinuous at the boundary, according to the surface-charge distribution. Thus:

$$\left[\frac{\partial V_2}{\partial r} - \frac{\partial V_1}{\partial r}\right]_{r=a} = -\frac{1}{\varepsilon_0}\sigma(\theta)$$

That is:

$$(B_\ell(-\ell-1)a^{-\ell-2} - A_\ell \ell a^{\ell-1})P_\ell(\cos\theta) = -\frac{1}{\varepsilon_0}\sigma(\theta)$$

Using our derived relation between the inner and outer coefficients, we end up with, after a little algebra:

$$A_\ell P_\ell(\cos\theta)(2\ell+1) = \frac{a^{1-\ell}}{\varepsilon_0}\sigma(\theta)$$

Now, let us find the coefficients. To do this, we multiply both sides by $P_m(\cos\theta)\sin\theta$ (the reason for the P_m is that of orthonormality; and that of the $\sin\theta$ will become clear when we integrate for orthonormality), and integrate over the Legendre polynomials period $0 \to \pi$. Thus:

$$A_\ell(2\ell+1)\int_0^\pi P_\ell(\cos\theta)P_m(\cos\theta)\sin\theta d\theta = \frac{a^{1-\ell}}{\varepsilon_0}\int_0^\pi \sigma(\theta)P_m(\cos\theta)\sin\theta d\theta$$

Let us consider the LHS. Let $x = \cos\theta \Rightarrow dx = -\sin\theta d\theta$. Then, also changing the limits, we have:

$$A_\ell(2\ell+1)\int_{-1}^1 P_\ell(x)P_m(x)dx$$

This integrates to, taking $\ell = m$:

$$A_\ell(2\ell+1)\int_{-1}^{1} P_\ell(x)P_m(x)dx = A_\ell(2\ell+1)\frac{2}{2\ell+1} = 2A_\ell$$

Hence, we have:

$$A_\ell = \frac{a^{1-\ell}}{2\varepsilon_0}\int_0^\pi \sigma(\theta)P_\ell(\cos\theta)\sin\theta d\theta$$

Now, let us insert our expression for the charge distribution (and the reason for expressing it so will become clear):

$$A_\ell = \frac{a^{1-\ell}\sigma_0}{2\varepsilon_0 3}\int_0^\pi [4P_2(\cos\theta) - P_0(\cos\theta)]P_\ell(\cos\theta)\sin\theta d\theta$$

Hence, we see that only $\ell = 0, 2$ will produce non-zero coefficients; due to the orthonormality of the Legendre polynomials. So, evaluating the $\ell = 0$ case:

$$\begin{aligned} A_0 &= -\frac{a\sigma_0}{6\varepsilon_0}\int_0^\pi P_0^2(\cos\theta)\sin\theta d\theta \\ &= -\frac{a\sigma_0}{6\varepsilon_0}2 \\ &= -\frac{a\sigma_0}{3\varepsilon_0} \end{aligned}$$

Where we have used the standard relation:

$$\int_0^\pi P_\ell^2(\cos\theta)\sin\theta d\theta = \frac{2}{2\ell+1}$$

Which is put into the usual form:

$$\int_{-1}^{1} P_\ell^2(x)dx = \frac{2}{2\ell+1}$$

By using the substitution $x = \cos\theta$. Which is the reason for multiplying by $\sin\theta$ previously. Let us compute the $\ell = 2$ case:

$$\begin{aligned} A_2 &= \frac{\sigma_0}{6a\varepsilon_0}4\int_0^\pi P_2^2(\cos\theta)\sin\theta d\theta \\ &= \frac{\sigma_0}{6a\varepsilon_0}4\frac{2}{5} \\ &= \frac{4\sigma_0}{15a\varepsilon_0} \end{aligned}$$

Now, let us write down the potential inside the shell:

$$\begin{aligned} V_1(r,\theta) &= \sum_\ell A_\ell r^\ell P_\ell(\cos\theta) \\ &= A_0P_0(\cos\theta) + A_2r^2P_2(\cos\theta) \\ &= A_0 + A_2r^2P_2(\cos\theta) \\ &= -\frac{a\sigma_0}{3\varepsilon_0} + \frac{4\sigma_0}{15\varepsilon_0 a}r^2P_2(\cos\theta) \\ &= -\frac{\sigma_0 a}{3\varepsilon_0}\left[1 - \frac{4r^2}{5a^2}P_2(\cos\theta)\right] \qquad r \le a \end{aligned}$$

And outside, using our derived relation between coefficients:

$$\begin{aligned} V_2(r,\theta) &= \sum_\ell a^{2\ell+1} r^{-\ell-1} A_\ell P_\ell(\cos\theta) \\ &= -\frac{\sigma_0 a}{3\varepsilon_0}\left[\frac{a}{r} - \frac{4a^3}{5r^3} P_2(\cos\theta)\right] \qquad r \geq a \end{aligned}$$

And therefore, we have, as the exact scalar potential:

$$\begin{aligned} V_1(r,\theta) &= -\frac{\sigma_0 a}{3\varepsilon_0}\left[1 - \frac{4r^2}{5a^2} P_2(\cos\theta)\right] \qquad r \leq a \\ V_2(r,\theta) &= -\frac{\sigma_0 a}{3\varepsilon_0}\left[\frac{a}{r} - \frac{4a^3}{5r^3} P_2(\cos\theta)\right] \qquad r \geq a \end{aligned}$$

B.3 Show That $T^{\mu\nu\lambda}$ Is a Tensor

From the relativistic electrodynamics section, we have the following definitions:

$$\begin{aligned} T^{\mu\nu\lambda} &\equiv \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} \\ F^{\mu\nu} &\equiv \partial^\mu A^\nu - \partial^\nu A^\mu \end{aligned}$$

And we have that the inverse transformation of a first rank contravariant tensor:

$$B^\mu = \frac{\partial x^\mu}{\partial x'^\nu} B'^\nu$$

Now, in the main text, we have shown that the field tensor transforms as a second rank contravariant tensor:

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} F^{\mu\nu}$$

And we have that the 4-current density transforms as a first rank tensor, as well as the contravariant derivative operator:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu \qquad \partial'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\partial^\nu$$

So, let us show that $T^{\mu\nu\lambda}$ transforms as a third rank contravariant tensor. That is, we must show that the following is satisfied:

$$T'^{\mu\nu\lambda} = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\rho}\frac{\partial x'^\lambda}{\partial x^\pi} T^{\sigma\rho\pi}$$

That is, expanding out the definition of $T^{\mu\nu\lambda}$:

$$\left(\partial'^\mu F'^{\nu\lambda} + \partial'^\nu F'^{\lambda\mu} + \partial'^\lambda F'^{\mu\nu}\right) = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\rho}\frac{\partial x'^\lambda}{\partial x^\pi}\left(\partial^\sigma F^{\rho\pi} + \partial^\rho F^{\pi\sigma} + \partial^\pi F^{\sigma\rho}\right)$$

Now, let us pick off the first term:

$$\frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\rho}\frac{\partial x'^\lambda}{\partial x^\pi}\partial^\sigma F^{\rho\pi}$$

Let us write the far right expressions in terms of their counterparts in the primed frame:

$$\frac{\partial x'^{\mu}}{\partial x^{\sigma}}\frac{\partial x'^{\nu}}{\partial x^{\rho}}\frac{\partial x'^{\lambda}}{\partial x^{\pi}}\frac{\partial x^{\sigma}}{\partial x'^{\alpha}}\frac{\partial x^{\rho}}{\partial x'^{\beta}}\frac{\partial x^{\pi}}{\partial x'^{\gamma}}\partial'^{\alpha}T'^{\beta\gamma}$$

We notice that most of this collapses into Kronecker deltas:

$$\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\lambda}_{\gamma}\partial'^{\alpha}T'^{\beta\gamma}$$

And that leaves us with:

$$\partial'^{\mu}T'^{\nu\lambda}$$

Which is clearly what we needed to prove; for this first term. It is tedious to do the same for the other two terms, and it is plain that they transform in the same way.

Hence proven.

B.4 Lorentz Force in Covariant Form

We wish to show that the Lorentz force and rate of change of energy of a particle, charge q; induced by electric & magnetic fields; can be written as:

$$\frac{dp^{\mu}}{d\tau} = qF^{\mu\nu}u_{\nu}$$

Let us start by discussing the rate of change of energy. Let us denote the energy of the particle by ϵ. Now, the work W done is scalar product of the force and displacement:

$$W = \boldsymbol{F}\cdot\boldsymbol{d}$$

Then the rate of change of energy is the rate of change of work:

$$\frac{dW}{dt} = \frac{d\epsilon}{dt} = \boldsymbol{F}\cdot\boldsymbol{v}$$

Now, the force on a charged particle is given by the Lorentz force law:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

And therefore:

$$\begin{aligned}\frac{d\epsilon}{dt} &= \boldsymbol{F}\cdot\boldsymbol{v}\\ &= q\boldsymbol{v}\cdot(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\\ &= q\boldsymbol{v}\cdot\boldsymbol{E} + q\boldsymbol{v}\cdot(\boldsymbol{v}\times\boldsymbol{B})\\ &= q\boldsymbol{E}\cdot\boldsymbol{v}\end{aligned}$$

Where we have used the standard vector identity that $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{a}) = 0$. We shall now start to take a different “tack” on the problem.

Now, let us state the following 4-vectors:

$$u^\mu = (c\gamma, \gamma\boldsymbol{u}) \qquad p^\mu = (\epsilon/c, \boldsymbol{p}) \qquad d\tau = \frac{dt}{\gamma}$$

And the field-tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix}$$

Now, notice:

$$\frac{dp^\mu}{d\tau} = \gamma\frac{dp^\mu}{dt} \qquad u_\mu = (c\gamma, -\gamma\boldsymbol{u})$$

Now, let us multilply the field tensor by the velocity 4-vector:

$$\begin{aligned} F^{\mu\nu}u_\nu &= \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix}\begin{pmatrix} c\gamma \\ -\gamma u_1 \\ -\gamma u_2 \\ -\gamma u_3 \end{pmatrix} \\ &= \begin{pmatrix} E_1\gamma u_1/c + E_2\gamma u_2/c + E_3\gamma u_3/c \\ E_1\gamma + B_3\gamma u_2 - B_2\gamma u_3 \\ E_2\gamma + B_2\gamma u_1 - B_1\gamma u_2 \\ E_3\gamma + B_2\gamma u_1 - B_1\gamma u_2 \end{pmatrix} \end{aligned}$$

Now, let us consider again the original equation:

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu$$

That is, the μ^{th} component of the LHS is the same as the μ^{th} component of the RHS. Let us write this out:

$$\begin{pmatrix} \frac{\gamma}{c}\frac{d\epsilon}{dt} \\ \gamma\frac{dp_1}{dt} \\ \gamma\frac{dp_2}{dt} \\ \gamma\frac{dp_3}{dt} \end{pmatrix} = q\begin{pmatrix} E_1\gamma u_1/c + E_2\gamma u_2/c + E_3\gamma u_3/c \\ E_1\gamma + B_3\gamma u_2 - B_2\gamma u_3 \\ E_2\gamma + B_2\gamma u_1 - B_1\gamma u_2 \\ E_3\gamma + B_2\gamma u_1 - B_1\gamma u_2 \end{pmatrix}$$

That is, equating components:

$$\begin{aligned} \frac{\gamma}{c}\frac{d\epsilon}{dt} &= q(E_1\gamma u_1/c + E_2\gamma u_2/c + E_3\gamma u_3/c) \\ \gamma\frac{dp_1}{dt} &= q(E_1\gamma + B_3\gamma u_2 - B_2\gamma u_3) \\ \gamma\frac{dp_2}{dt} &= q(E_2\gamma + B_2\gamma u_1 - B_1\gamma u_2) \\ \gamma\frac{dp_3}{dt} &= q(E_3\gamma + B_2\gamma u_1 - B_1\gamma u_2) \end{aligned}$$

Cancelling off various factors:

$$\begin{aligned}
\frac{d\epsilon}{dt} &= q(E_1 u_1 + E_2 u_2 + E_3 u_3) \\
\frac{dp_1}{dt} &= q(E_1 + B_3 u_2 - B_2 u_3) \\
\frac{dp_2}{dt} &= q(E_2 + B_2 u_1 - B_1 u_2) \\
\frac{dp_3}{dt} &= q(E_3 + B_2 u_1 - B_1 u_2)
\end{aligned}$$

The latter 3 equations are components of a cross-product; and the first of a scalar product. Finally, notice:

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \qquad \frac{d\epsilon}{dt} = \boldsymbol{v} \cdot \boldsymbol{E}$$

Examination of what we have from unpacking our notation, and what we have above, will reveal that the following equation has, embedded inside, both the Lorentz force law & the rate of change of energy of a particle:

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu} u_\nu \tag{B.1}$$

Thanks to J.Agnew for helping out with this derivation!

C Legendre Equation & Spherical Harmonics

In spherical polars (i.e. the (r, θ, ϕ) coordinate system), have have that the Laplace equation $\nabla^2 \Phi = 0$ has the form:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{C.1}$$

Now, let us assume that we can solve the equation using the standard 'separation of variables' technique. Thus:

$$\Phi(r, \theta, \phi) = \frac{U(r)}{r}P(\theta)Q(\phi)$$

Where the usual $R(r)$ is obviously just $R(r) = \frac{U(r)}{r}$. Hence, we have that:

$$PQ\frac{d^2U}{dr^2} + \frac{1}{r^2\sin\theta}UQ\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} + \frac{1}{r^2\sin^2\theta}UP\frac{d^2Q}{d\phi^2} = 0$$

Let us collect some terms together, and multiply by $\frac{r^2\sin^2\theta}{RPQ}$; giving:

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{Pr^2\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta}\right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0$$

Now, we can assign constant terms to this equation, so that:

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \tag{C.2}$$

Which, semi-trivally, has solutions:

$$Q_m(\phi) = e^{\pm im\phi} \tag{C.3}$$

So, we have:

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{Pr^2\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta}\right] - m^2 = 0$$

This is just:

$$\frac{r^2}{U}\frac{d^2U}{dr^2} + \frac{1}{P\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} - \frac{m^2}{\sin^2\theta} = 0$$

Again, let us use another constant term:

$$\frac{r^2}{U}\frac{d^2U}{dr^2} = \ell(\ell+1) \tag{C.4}$$

Then we have:

$$\ell(\ell+1) + \frac{1}{P\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} - \frac{m^2}{\sin^2\theta} = 0$$

That is:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

Now, this is more conventionally written, with $x \equiv \cos\theta$, as:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0 \tag{C.5}$$

And is known as the *generalised Legendre equation*, solutions of which are known as the *associated Legendre polynomials*.

C.0.1 Radial Solution

Let us quickly derive the solution to the radial part of the Laplace equation. That is, (C.4). So, we have:

$$\frac{r^2}{U}\frac{d^2U}{dr^2} = \ell(\ell+1)$$

Let us suppose that the solution is a power law. Let us try $U(r) = r^\alpha$. Then:

$$\begin{aligned}\frac{r^2}{r^\alpha}\frac{d^2}{dr^2}(r^\alpha) &= r^2 r^{-\alpha}\alpha(\alpha-1)r^{\alpha-1} \\ &= \alpha(\alpha-1)\end{aligned}$$

So, we have that:

$$\alpha(\alpha-1) = \ell(\ell+1) \quad \Rightarrow \quad \alpha^2 - \alpha - \ell^2 - \ell = 0$$

Now, its not immediately obvious how to solve this; but, if the following expression is written, we see that it is equivalent:

$$(\alpha+\ell)[\alpha-\ell-1] = \alpha^2 - \alpha - \ell^2 - \ell$$

Hence equivalent. Also, by writing in this form, we are able to see its solutions:

$$(\alpha+\ell)[\alpha-\ell-1] = 0$$

That is, two solutions:

$$\alpha = -\ell \qquad \alpha = \ell+1$$

Hence, we have that the radial solution is of the form (putting in arbitrary constants):

$$U(r) = Ar^{\ell+1} + Br^{-\ell} \tag{C.6}$$

We have also used the form $R(r)$; hence, it is:

$$R(r) = \frac{U(r)}{r} = Ar^\ell + Br^{-\ell-1}$$

C.1 Power Series Solution to the Ordinary Legendre Equation

Now, in (C.5), we had a dependance upon m; that is, something that was not azimuthally symmetric. Let us consider the case where $m = 0$. Then, the resulting equation is known as the *ordinary Legendre differential equation*:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0 \tag{C.7}$$

Now, recall that $x = \cos\theta$; where we have that $\theta \in [0,\pi]$; so that the function must be well behaved on $-1 \le x \le 1$. Now, we will assume that the solution takes on a power series form:

$$P(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j = a_j x^{j+\alpha}$$

So, calculating the necessary expressions (suppressing the summation signs):

$$\begin{aligned}
\frac{dP}{dx} &= x^\alpha a_j j x^{j-1} + \alpha x^{\alpha-1} a_j x^j \\
&= a_j j x^{\alpha+j-1} + \alpha a_j x^{\alpha+j-1} \\
&= (a_j j + \alpha a_j) x^{\alpha+j-1} \\
\Rightarrow (1-x^2)\frac{dP}{dx} &= (1-x^2)(a_j j + \alpha a_j) x^{\alpha+j-1} \\
&= (a_j j + \alpha a_j)[x^{\alpha+j-1} - x^{\alpha+j+1}] \\
\Rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] &= a_j (j+\alpha)[(\alpha+j-1)x^{\alpha+j-2} - (\alpha+j+1)x^{\alpha+j}]
\end{aligned}$$

Thus, the total equation is:

$$a_j(j+\alpha)[(\alpha+j-1)x^{\alpha+j-2} - (\alpha+j+1)x^{\alpha+j}] + \ell(\ell+1)a_j x^{j+\alpha} = 0$$

Collecting like-powers of x, and putting summations back in:

$$\sum_{j=0}^{\infty} a_j(j+\alpha)(\alpha+j-1)x^{\alpha+j-2} - \sum_{j=0}^{\infty} a_j[(j+\alpha)(\alpha+j+1) - \ell(\ell+1)]x^{j+\alpha} = 0$$

Now, we apply a standard 'trick' to solving these things further. In the first expression, change summation index from j to $m = j-2$. Then, the first expression becomes:

$$\sum_{m=2}^{\infty} a_{m+2}(m+2+\alpha)(\alpha+m+1)x^{\alpha+m}$$

Then, this will give us a recursion relation:

$$a_{j+2} = \frac{(\alpha+j)(\alpha+j+1) - \ell(\ell+1)}{(\alpha+j+1)(\alpha+j+2)} a_j$$

From this, and the requirement that the series remain finite, we see that there is a cut-off. We only have the cases for $\alpha = 0, 1$ that terminate, and they produce even, odd polynomials. Essentially, it will give us the Legendre polynomials, of order ℓ:

$$P_)(x) = 1 \quad P_1(x) = x \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1) \quad P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

Notice that each polynomial possesses only even or odd power; not both. This is down to the recursive relation; and that the series must converge for $|x| \leq 1$. We also have that $\ell \geq 0$, and integer. By a manipulation of the power series, we are able to (but is not done here) find a generation function: the *Rodrigues' formula*:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \tag{C.8}$$

C.1.1 Orthogonality of Legendre Polynomials

Now, let us write down the Legendre equation again:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell(x)}{dx}\right]+\ell(\ell+1)P_\ell(x)=0$$

Let us multiply the equation by $P_{\ell'}(x)$, and integrate over the interval:

$$\int_{-1}^{1}P_{\ell'}(x)\left\{\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell(x)}{dx}\right]+\ell(\ell+1)P_\ell(x)\right\}dx=0$$

Multiplying through:

$$\int_{-1}^{1}P_{\ell'}(x)\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell(x)}{dx}\right]+\ell(\ell+1)P_{\ell'}(x)P_\ell(x)dx=0$$

Now, integrating the first expression, by parts:

$$\begin{aligned}\int_{-1}^{1}P_{\ell'}(x)\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell(x)}{dx}\right]dx &= P_{\ell'}(1-x^2)\frac{dP_\ell}{dx}\bigg|_{-1}^{1}-\int_{-1}^{1}(1-x^2)\frac{dP_\ell}{dx}\frac{dP_{\ell'}}{dx}dx\\ &= \int_{-1}^{1}(x^2-1)\frac{dP_\ell}{dx}\frac{dP_{\ell'}}{dx}dx\end{aligned}$$

As we have that the series terminates at ±1. Putting this back in:

$$\int_{-1}^{1}(x^2-1)\frac{dP_\ell}{dx}\frac{dP_{\ell'}}{dx}+\ell(\ell+1)P_{\ell'}(x)P_\ell(x)dx=0$$

Now, write this down again, but with $\ell\to\ell'$; then subtract. This is easily shown to be just:

$$[\ell(\ell+1)-\ell'(\ell'+1)]\int_{-1}^{1}P_{\ell'}(x)P_\ell(x)dx=0 \tag{C.9}$$

Which says that if $\ell\neq\ell'$, then the integral must vanish. However, to calculate the integral for the $\ell=\ell'$ (i.e. its 'square') then we must actually compute the integral, using the explicit form of the Legendre polynomials; that is, using Rodrigues' formula. So, for $\ell=\ell'$, the integral is therefore:

$$\begin{aligned}N_\ell &\equiv \int_{-1}^{1}P_\ell^2(x)dx\\ &= \frac{1}{2^{2\ell}(\ell!)^2}\int_{-1}^{1}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\frac{d^\ell}{dx^\ell}(x^2-1)^\ell dx\end{aligned}$$

After some non-trivial integration (which is not repeated here), we are able to get to the standard orthogonality condition:

$$\int_{-1}^{1}P_{\ell'}(x)P_\ell(x)dx=\frac{2}{2\ell+1}\delta_{\ell\ell'} \tag{C.10}$$

C.1.2 Using Legendre Polynomials as a Basis

Now, as the Legendre polynomials form a complete set of orthogonal functions, any function on $[-1, 1]$ can be expressed in terms of them; thus:

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x)$$

To find the coefficients a_ℓ, we apply a standard 'fourier-type' technique:
Multiply both sides by another polynomial, and integrate:

$$\int_{-1}^{1} P_{\ell'}(x) f(x) dx = \sum_{\ell=0}^{\infty} a_\ell \int P_\ell(x) P_{\ell'}(x) dx$$

Now, the integral on the RHS is just the orthogonality relation. Thus:

$$\int_{-1}^{1} P_{\ell'}(x) f(x) dx = \sum_{\ell} a_\ell \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

Where the summation will just pick out a single value, ℓ', as being no-zero. Hence:

$$\int_{-1}^{1} P_{\ell'}(x) f(x) dx = a_{\ell'} \frac{2}{2\ell'+1}$$

We can, of course, relabel the indices, back to ℓ. Hence:

$$a_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} P_\ell(x) f(x) dx$$

C.2 Associated Legendre Polynomials & Spherical Harmonics

Here, we consider the solutions to (C.5), for $m \neq 0$; that is, for problems without azimuthal symmetry. So, we must solve the equation for arbitrary ℓ, m. We basicaly have the generalisation of the Legendre polynomial $P_\ell(\cos\theta)$; namely, the *associated Legendre polynomial* $P_\ell^m(x)$. The version of Rodrigues formula here is now:

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \quad \text{(C.11)}$$

Now, it can be shown that:

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x) \quad \text{(C.12)}$$

In a similar fashion as we used previously, we have an orthogonality relation:

$$\int_{-1}^{1} P_{\ell'}^m(x) P_\ell^m(x) dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'} \quad \text{(C.13)}$$

Now, recall that we had Q_m as the azimuthal solution, (C.3). Now, they form a complete set on $\phi \in [0, 2\pi]$. Thus, as we have that $P_\ell^m(\cos\theta)$ form a complete set on $\theta \in [0, \pi]$, we can imagine that some combination will give us full solutions, on the unit sphere. We call these solutions *spherical harmonics*, and are given by the normalised product:

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\phi} \tag{C.14}$$

So, from (C.12), we can see that:

$$Y_{\ell -m}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi) \tag{C.15}$$

The normalisation/orthogonality relation is:

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} Y_{\ell' m'}^*(\theta, \phi) Y_{\ell m}(\theta, \phi) \sin\theta d\theta d\phi = \delta_{\ell\ell'}\delta_{mm'} \tag{C.16}$$

Note, for $m = 0$, we just have:

$$Y_{\ell 0}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$$

An arbitrary function may be expanded in terms of spherical harmonics, thus:

$$f(\theta, \phi) = \sum_\ell \sum_m a_{\ell m} Y_{\ell m}(\theta, \phi) \tag{C.17}$$

Where the coefficients are found, in a similar way to before; to be:

$$a_{\ell m} = \int Y_{\ell m}^*(\theta, \phi) f(\theta, \phi) \sin\theta d\theta d\phi$$

Just as way of convenient notation, we use the solid angle element, $d\Omega \equiv \sin\theta d\theta d\phi$, so that:

$$a_{\ell m} = \int Y_{\ell m}^*(\theta, \phi) f(\theta, \phi) d\Omega$$

C.3 The Addition Theorem for Spherical Harmonics

Consider two coordinate vectors, $\boldsymbol{x}, \boldsymbol{x}'$ with coordinates $(r, \theta, \phi), (r', \theta', \phi')$. The addition theorem states:

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi)$$

Where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$.
To prove the theorem, consider that the above Legendre polynomial may be expanded in terms of spherical harmonics, thus:

$$P_\ell(\cos\gamma) = \sum_{\ell'} \sum_m A_{\ell' m}(\theta', \phi') Y_{\ell' m}(\theta, \phi)$$

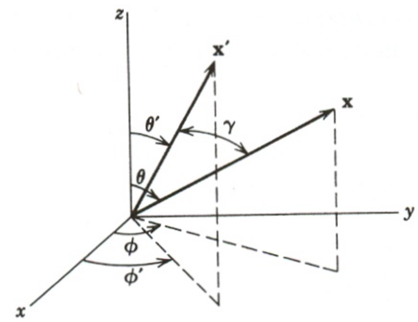

Figure 11: The addition theorem, for spherical polars. Figure from *Jackson*

By various symmetry arguements, we see that we have only the case where $\ell = \ell'$ in the sum. Thus:

$$P_\ell(\cos\gamma) = \sum_m A_m(\theta', \phi') Y_{\ell' m}(\theta, \phi)$$

Where the coefficients are given by (found in the usual way):

$$A_m(\theta', \phi') = \int Y^*_{\ell m}(\theta, \phi) P_\ell(\cos\gamma) d\Omega$$

C.4 Some Spherical Harmonics

Below are some spherical harmonics $Y_{\ell m}(\theta, \phi)$:
$\ell = 0$

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \tag{C.18}$$

$\ell = 1$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \tag{C.19}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \tag{C.20}$$

$\ell = 2$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\phi} \tag{C.21}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{i\phi} \tag{C.22}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} (\tfrac{1}{2}\cos^2\theta - 1) \tag{C.23}$$

$\ell = 3$

$$Y_{33} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta e^{3i\phi} \quad (C.24)$$

$$Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta e^{2i\phi} \quad (C.25)$$

$$Y_{31} = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta(5\cos^2\theta - 1)e^{i\phi} \quad (C.26)$$

$$Y_{30} = \sqrt{\frac{7}{4\pi}}(\tfrac{5}{2}\cos^3\theta - \tfrac{3}{2}\cos\theta) \quad (C.27)$$

Note, we have previously written:

$$Y_{\ell -m} = (-1)^m Y^*_{\ell m} \quad (C.28)$$

Which gives a way of generating the unwritten negative m harmonics above. Also, remember, for a given ℓ, there are $2\ell + 1$ different spherical harmonics:

$$m = \ell, \ell - 1, \ldots, 0, \ldots, -\ell + 1, -\ell$$

Also note, that a spherical harmonic, with $m = 0$ is just the normalisation constant, multiplied by the Legendre polynomial:

$$Y_{\ell 0}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} P_\ell(\theta) \quad (C.29)$$

C.5 Generating Function

For the Legendre polynomials, we suppose that a function will generate the polynomials, then we will check that it does:

$$f(x, t) = \sum_{n=0}^{\infty} t^n P_n(x) \quad (C.30)$$

$$= \frac{1}{\sqrt{1 - 2tx + t^2}} \quad (C.31)$$

Now, let us find both the x- and t-derivatives of this expression:

$$\frac{d}{dx} : \quad \sum_{n=0}^{\infty} t^n P_n'(x) = t(1 - 2tx + t^2)^{-3/2} \quad (C.32)$$

$$\frac{d}{dt} : \quad \sum_{n=0}^{\infty} n t^{n-1} P_n(x) = (x - t)(1 - 2tx + t^2)^{-3/2} \quad (C.33)$$

Now, let us express the first in the following way:

$$\begin{aligned} t(1 - 2tx + t^2)^{-3/2} &= t(1 - 2tx + t^2)^{-1}(1 - 2tx + t^2)^{-1/2} \\ &= t(1 - 2tx + t^2)^{-1} \sum_n t^n P_n(x) \end{aligned}$$

Hence:

$$\sum_n t^n P_n'(x) = t(1-2tx+t^2)^{-1}\sum_n t^n P_n(x)$$

That is:

$$t\sum_n t^n P_n(x) = (1-2tx+t^2)\sum_n t^n P_n'(x)$$

Which is just:

$$\sum_n t^{n+1} P_n(x) = \sum_n (t^n - 2xt^{n+1} + t^{n+2}) P_n'(x)$$

Solve solve this (by equating coefficients of like powers of t), we note the following:

$$\sum_n (t^n - 2xt^{n+1} + t^{n+2}) P_n'(x) = \sum_n t^n P_n'(x) - 2x\sum_n t^{n+1} P_n'(x) + \sum_n t^{n+2} P_n'(x)$$

Where, in each sum above, n starts from zero. Now, we may relabel each summation variable thus:

$$\Rightarrow \quad t^{n+1} P_{n+1}'(x) - 2xt^{n+1} P_n'(x) + t^{n+1} P_{n-1}'(x)$$

Hence, now we equate powers of t, giving:

$$P_n(x) = P_{n+1}'(x) - 2xP_n'(x) + P_{n-1}'(x) \tag{C.34}$$

Now, let us go back a bit. If we mutiply (C.32) by $(x-t)$, and (C.33) by t, then they should be equal. Hence:

$$\sum_n (x-t)t^n P_n'(x) = \sum_n nt^n P(x)$$

That is:

$$xt^n P_n' - t^{n+1} P_n' - nt^n P_n = 0$$

Again, shifting the summation variable, to be able to equate coefficients:

$$xt^n P_n' - t^n P_{n-1}' - nt^n P_n = 0$$

Hence:

$$xP_n' - P_{n-1}' = nP_n \tag{C.35}$$

Now, let us write down (C.34) and (C.35), together:

$$\begin{aligned} P_n &= P_{n+1}' - 2xP_n' + P_{n-1}' \\ nP_n &= xP_n' - P_{n-1}' \end{aligned}$$

From these, eliminate P_{n-1}', giving:

$$(n+1)P_n = P_{n+1}' - xP_n'$$

Now, in the above, let $n \to n+1$:

$$nP_{n-1} = P_n' - xP_{n-1}'$$

Now, add (C.35) times x to the above; giving:

$$x^2 P_n' - xP_{n-1}' + nP_{n-1} = nxP_n + P_n' - xP_{n-1}'$$

Tidying up:

$$(1 - x^2)P_n' = n(P_{n-1} - xP_n)$$

Differentiate both sides, with respect to x:

$$[(1 - x^2)P_n']' = n(P_{n-1}' - P_n - xP_n')$$

Now, using (C.35) again: $xP_n' - P_{n-1}' = nP_n$, on the RHS

$$[(1 - x^2)P_n']' = -n(n + 1)P_n$$

Hence, we have shown:

$$[(1 - x^2)P_m'(x)]' = -m(m + 1)P_m(x)$$

Hence completing the proof.

Let us compute $f(1, t)$:

$$\begin{aligned} f(1,t) &= \sum_{n=0}^{\infty} P_n(1)t^n \\ &= \frac{1}{(1 - 2tx + t^2)^{1/2}} \\ &= \frac{1}{1 - t} \\ &= \sum_n t^n \end{aligned}$$

Hence, we see that $P_n(1) = 1$.

C.5.1 Application: Expand $\frac{1}{|r_1 - r_2|}$

We often need to expand:

$$\frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}$$

Now, we can do so in terms of orthogonal functions, in the angle between the two vectors. Let us assume $r_1 > r_2$. So:

$$\begin{aligned} \frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} &= \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}} \\ &= \frac{1}{r_1} \frac{1}{\sqrt{1 + (\frac{r_2}{r_1})^2 - 2\frac{r_2}{r_1}\cos\theta}} \end{aligned}$$

Now, we identify the expression with the generating function of the Legendre polynomials, as:

$$t = \frac{r_2}{r_1} \qquad x = \cos\theta$$

Hence, we have:

$$\frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = \frac{1}{r_1} \sum_{n=0}^{\infty} \left(\frac{r_2}{r_1} \right)^n P_n(\cos\theta) \tag{C.36}$$

For $r_2 > r_1$. We can obviously rearrange the indices if the reverse is true.

D Books

There are a number of very useful texts, regarding this topic of Electrodynamics. Throughout the course of writing this, I have consulted many of the below books.

- *Heald & Marion*: Classical Electromagnetic Radiation;
- *Jackson*: Classical Electrodynamics;
- *Landau & Lifshitz*: The Classical Theory of Fields;
- *Shutz*: A First Course in General Relativity;
- *Zwiebach*: A First Course in String Theory;
- *Woodhouse*: Special Relativity.

Heald & Marion is a very good text for many of the electrostatic problems, but is very light on the relativistic part of the subject. Jackson is a classic text, designed for graduate courses in the subject, so goes way beyond the scope of the current treatise; regardless, it is very good for the multipole expansion, and especially in spherical harmonic theory.
The final 4 books were consulted mainly for tensor theory. Zwiebach is an excellent start point for Lorentz transformations & simple tensor manipulations; but is quite light if one is looking for a more formal introduction. Hence, consulting Schutz is heavily recommended; infact, most introductory general relativity texts tend to have a large section devoted to tensor theory & very helpful examples & hints for manipulations. Schutz introduces tensors in a very formal manner, but grounds it firmly in physical ideas with plently of examples. Woodhouse is a very mathematical text (whereas the others are firmly grounded in physics), introducing special relativity in a very abstract manner.

The, frankly, classical theoretical physics series by Landau & Lifshitz are highly recommended. The book relevant for electrodynamics is their Vol 2. It has an excellent section on the non-relativistic part; going into a lot of detail. The latter half of the book is on general relativity, so is well worth looking at, as it introduces tensors very well.