

# Advanced Quantum Mechanics: Quick Guide

J.Pearson\*

(Dated: January 28, 2009)

## Abstract

This is a quick guide – a summary – of the Advanced Quantum Mechanics course at the University of Manchester, taught by S.Grigorenko between Sept '08 and Dec '08. These summary notes are based upon his lecture notes. A copy of my full lecture notes, on this topic, may be found at [www.jpoffline.com](http://www.jpoffline.com).

Keywords:

---

\*Electronic address: [jon@jpoffline.com](mailto:jon@jpoffline.com)

## Contents

I. Basic Formalism	3
II. Formalism of Vector Spaces	3
III. Many Body Problems	7
IV. Symmetry in Quantum Mechanics	8
V. Angular Momentum	9
VI. Charged Particles in EM Fields	10

## I. BASIC FORMALISM

The approaches we have considered are:

- In **Copenhagen** formalism the objects we work with are the **wavefunctions**  $\psi(x)$ , in **coordinate-space**. The **Schrodinger equation** is  $i\hbar\dot{\psi} = \hat{\mathcal{H}}\psi$ . This approach is good for one-body problems, but pretty much fails when including constraints, and for many-body problems.
- **Dirac** formalism has the objects being **abstract vectors**  $|\psi\rangle$  in **Hilbert space**. This approach is good for many-body problems; but fails upon inclusion of constraints.
- The **path integral** approach is good when working with **fields**, but is mathematically vulnerable.

It should be noted that all formalisms are essentially the same, but use different tools to get a result.

The **Schrodinger picture** of quantum mechanics has the states of a system evolving, with operators having no explicit time dependence.

The **Heisenberg picture** has operators evolving, and states staying constant; with the equation of motion

$$\frac{d}{dt}\hat{Q}_{\mathcal{H}}(t) = \frac{\partial\hat{Q}}{\partial t} - \frac{i}{\hbar} [\hat{Q}, \hat{\mathcal{H}}].$$

## II. FORMALISM OF VECTOR SPACES

In the **coordinate basis**  $|x\rangle$ , we expand a wavefunction in **ket-space**,

$$|\psi\rangle = \sum_x \psi_x |\psi\rangle,$$

where we make the identification

$$\psi_x = \langle x|\psi\rangle = \psi(x).$$

Similarly, we expand in **bra-space**,

$$\langle\psi| = \sum_x \psi_x^* \langle x|.$$

The coordinate basis is **orthonormal**,

$$\langle x|x'\rangle = \delta_{xx'}.$$

By forming the scalar product of two states, we see that

$$\begin{aligned}\langle \chi|\psi\rangle &= \sum_{x,x'} \chi_{x'}^* \langle x'|x\rangle \psi_x \\ &= \sum_x \chi_x^* \psi_x \\ &= \int dx \chi^*(x) \psi(x).\end{aligned}$$

The **completeness of the basis** is the statement that

$$\sum_x |x\rangle\langle x| = 1.$$

If instead of using the coordinate basis, we use the **orthonormal**  $|n\rangle$ -basis, then everything still holds:

$$|\psi\rangle = \sum_n a_n |n\rangle \quad \Rightarrow \quad a_n = \langle n|\psi\rangle \quad \Rightarrow \quad \sum_n |n\rangle\langle n| = 1.$$

**Operators** are linear,  $|\chi\rangle = \hat{Q}|\psi\rangle$ . An operator can be expressed in a basis by inserting the unity operator,

$$\begin{aligned}\hat{Q} &= 1.\hat{Q}.1 \\ &= \sum_{m,n} |m\rangle\langle m|\hat{Q}|n\rangle\langle n| \\ &= \sum_{m,n} Q_{mn} |m\rangle\langle n|.\end{aligned}$$

We have arrived at the **matrix representation** of the operator

$$Q_{mn} = \langle m|\hat{Q}|n\rangle.$$

In a different basis, the matrix representation is

$$\begin{aligned}Q_{\tilde{m}\tilde{n}} &= \langle \tilde{m}|\hat{Q}|\tilde{n}\rangle \\ &= \sum_{m,n} \langle \tilde{m}|m\rangle \langle m|\hat{Q}|n\rangle \langle n|\tilde{n}\rangle \\ &= \sum_{m,n} Q_{mn} \langle \tilde{m}|m\rangle \langle n|\tilde{n}\rangle,\end{aligned}$$

which bears resemblance to tensor transformations.

Operators in **their basis** are simple,

$$\hat{x}|x\rangle = x|x\rangle, \quad \hat{p}|p\rangle = p|p\rangle.$$

To get an eigenstate in a different basis, one projects onto the basis required. For example, the momentum eigenstate, in coordinate representation:

$$\begin{aligned} \hat{p}|p\rangle &= p|p\rangle \\ \Rightarrow \langle x|\hat{p}|p\rangle &= p\langle x|p\rangle \\ \Rightarrow \sum_{x'} \langle x|\hat{p}|x'\rangle \langle x'|p\rangle &= p\langle x|p\rangle \\ \Rightarrow \sum_{x'} \langle x|\hat{p}|x'\rangle P(x') &= pP(x) \\ \Rightarrow -i\hbar \frac{d}{dx} P(x) &= pP(x) \\ \Rightarrow P(x) &= \frac{1}{(\sqrt{2\pi\hbar})^n} e^{\frac{ipx}{\hbar}} \end{aligned}$$

The **properties of operators** are

- **Hermitian:**  $\hat{Q}^\dagger = \hat{Q}$ . Such operators have real eigenvalues.
- **Anti-Hermitian:**  $\hat{Q}^\dagger = -\hat{Q}$ . One can make an anti-Hermitian operator Hermitian, by multiplying by  $i$ .
- **Unitary:**  $\hat{U}^\dagger \hat{U} = 1$ .
- The **eigenvalue equation** is such that  $\hat{Q}|\psi\rangle = \lambda|\psi\rangle$ , whereby states are normalised  $\langle\psi|\psi\rangle = 1$ .

**Raising and lowering operators** for the Harmonic oscillator have the following relations:

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= 1 \\ \hat{\mathcal{H}} &= \hbar\omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \end{aligned}$$

The **vacuum state** is such that  $\hat{a}|0\rangle = 0$ . In **coordinate representation**, one has

$$\sum_{x'} \langle x|\hat{a}|x'\rangle \langle x'|0\rangle = 0 \quad \Rightarrow \quad \sum_{x'} a_{xx'} \psi_0(x') = 0.$$

Furthermore, as  $\hat{a} = \alpha\hat{p} + \beta\hat{x}$ , then

$$a_{xx'} = \alpha \left( -i\hbar \frac{d}{dx} \right) \delta_{xx'} + \beta x \delta_{xx'},$$

which fairly easily allows one to find that the vacuum state, in coordinate representation, is

$$\psi_0(x) = C.e^{-\frac{m\omega x^2}{2\hbar}}.$$

In **momentum representation**, one has

$$\sum_{p'} \langle p|\hat{a}|p'\rangle \langle p'|0\rangle = 0 \quad \Rightarrow \quad \sum_{p'} a_{pp'} \psi_0(p') = 0.$$

To find  $\psi_0(p)$ , we must compute  $x_{pp'}$ , as

$$a_{pp'} = p\delta_{pp'} - im\omega x_{pp'}.$$

So, the position operator, in momentum representation:

$$\begin{aligned} x_{pp'} &= \langle p|\hat{x}|p'\rangle \\ &= \sum_{x,x'} \langle p|x\rangle \langle x|\hat{x}|x'\rangle \langle x'|p'\rangle \\ &= \sum_x x \langle p|x\rangle \langle x|p'\rangle \\ &= \int dx x P^*(x) P'(x) \\ &= C \int dx x e^{\frac{ix}{\hbar}(p'-p)} \\ &= D \frac{d}{dp'} \int dx e^{\frac{ix}{\hbar}(p'-p)} \\ &= i\hbar \delta_{pp'} \frac{d}{dp'}. \end{aligned}$$

Therefore, the vacuum state, in momentum representation:

$$p\psi_0(p) = -\hbar m\omega \frac{d}{dp} \psi_0(p) \quad \Rightarrow \quad \psi_0(p) = e^{-\frac{p^2}{2m\omega\hbar}}.$$

The **evolution operator** is

$$\hat{U}(t) = \sum_n e^{-\frac{iE_n t}{\hbar}} |n\rangle \langle n|$$

### III. MANY BODY PROBLEMS

**Bosons** are **symmetric** under exchange of two particles,

$$\psi(x_1, x_2, \dots) = \psi(x_2, x_1, \dots).$$

**Fermions** are **anti-symmetric** under the exchange

$$\psi(x_1, x_2, \dots) = -\psi(x_2, x_1, \dots).$$

The state  $|n_0, n_1, n_2 \dots\rangle$  tells us about the number of particles in a particular state, whereby  $n_k$  is the **occupancy number** of state  $k$ . The total number of particles in a system is then just

$$N = \sum_{i=0}^{\infty} n_i.$$

The different states with different  $N$  produce the **Fock space**.

The action of the **creation and annihilation** operators on **Bosons** are such that

$$\begin{aligned}\hat{a}_i^+ |n_0, \dots, n_i, \dots\rangle &= \sqrt{n_i + 1} |n_0, \dots, n_i + 1, \dots\rangle, \\ \hat{a}_i |n_0, \dots, n_i, \dots\rangle &= \sqrt{n_i} |n_0, \dots, n_i - 1, \dots\rangle.\end{aligned}$$

**Fermions** are the same, but have a transposition sign-factor upfront.

**Commutators** for **Bosons** are such that the **only non-zero** commutator is

$$[\hat{a}_i^+, \hat{a}_j] = \delta_{ij}.$$

**Fermions** have **anti-commutators**, whereby the **only non-zero** anti-commutator is

$$\{\hat{a}_i^+, \hat{a}_j\} = \delta_{ij}.$$

States may be generated from the vacuum, by applying the creation operator many times;

$$|n_0, n_1, \dots\rangle = \prod_j \frac{(\hat{a}_j^+)^{n_j}}{\sqrt{n_j!}} |0\rangle.$$

The Hamiltonian, under **secondary quantisation** is

$$\hat{\mathcal{H}} = \sum_n \varepsilon_n \hat{a}_n^+ \hat{a}_n + \frac{1}{2} \sum_{i,j,n,m} V_{ij,nm} \hat{a}_i^+ \hat{a}_j^+ \hat{a}_n \hat{a}_m,$$

where the **number of states** operator

$$\hat{n}_i = \hat{a}_i^+ \hat{a}_i,$$

and returns the number of particles in state  $i$ . This secondary quantisation is good for many-body problems.

#### IV. SYMMETRY IN QUANTUM MECHANICS

The symmetry operator commutes with the Hamiltonian,

$$[\hat{\mathcal{H}}, \hat{O}] = 0.$$

As they commute, the symmetry operator and Hamiltonian share a basis in which they are diagonal. Hence,

$$\frac{d}{dt}\langle \hat{O} \rangle = 0,$$

which means that  $\langle \hat{O} \rangle$  is **conserved** and quantum numbers associated with  $\hat{O}$  are **good quantum numbers**. If the symmetry operator is Hermitian, then there is a conserved measurable quantity.

The **translation operator** has a few representations,

$$\hat{T}_a = \sum_x |x\rangle\langle x+a|, \quad (\hat{T}_a)_{xx'} = e^{a\frac{d}{dx}}\delta_{xx'}, \quad (\hat{T}_a)_{xx'} = \delta(x+a-x).$$

The translation operator is unitary,

$$\hat{T}_a^\dagger \hat{T}_a = 1.$$

Expanding the translation operator, for small  $a$ ,

$$\hat{T}_{\delta a} = 1 + \hat{U}\delta a, \quad \hat{U} = \left. \frac{\partial \hat{T}_{\delta a}}{\partial a} \right|_{a=0}.$$

Directly following from the unitarity of  $\hat{T}_a$ , one can show that

$$\hat{U}^\dagger = -\hat{U},$$

so that  $\hat{R} = -i\hbar\hat{U}$  is unitary and Hermitian. Hence,

$$\hat{R} = -i\hbar \left. \frac{\partial \hat{T}}{\partial a} \right|_{a=0}.$$

$\hat{R}$  is an operator that describes a conserved symmetry quantity, and will provide good quantum numbers.

## V. ANGULAR MOMENTUM

The **total angular momentum** operator is

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}.$$

The **eigenstates** of angular momentum are

$$\hat{J}^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle, \quad \hat{J}_z|j, m\rangle = m\hbar|j, m\rangle.$$

**Commutators** are such that

$$[\hat{J}_a, \hat{J}_b] = i\hbar\hat{J}_c \epsilon_{abc}.$$

**Ladder operators** are defined such that

$$\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y,$$

and that their action on a state is

$$\hat{J}_{\pm}|j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\hbar|j, m \pm 1\rangle.$$

**Spin operators** are

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}},$$

where the **Pauli-spin matrices** are

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli matrices can be shown to conform to the commutation and anti-commutation relations:

$$[\hat{\sigma}_a \hat{\sigma}_b] = 2i\sigma_c \epsilon_{abc}, \quad \{\hat{\sigma}_a \hat{\sigma}_b\} = 2\delta_{ab}.$$

## VI. CHARGED PARTICLES IN EM FIELDS

The **Pauli Hamiltonian** is

$$\hat{\mathcal{H}} = \frac{1}{2m} \left( \hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right)^2 + q\phi - \frac{gq}{2mc} \hat{\mathbf{S}} \cdot \mathbf{B}.$$

If  $\mathbf{B}$  is constant, one can extract the **spin equation**,

$$i\hbar|\dot{\eta}\rangle = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B}|\eta\rangle, \quad \hat{\boldsymbol{\mu}} = \frac{gq}{2mc} \hat{\mathbf{S}}.$$

For electrons,

$$g = 2, \quad \hat{\boldsymbol{\mu}} = \mu_B \hat{\boldsymbol{\sigma}}.$$

The **phase shift**, due to magnetic potential, is

$$\varphi = \frac{q}{\hbar c} \oint_{\gamma} \mathbf{A} \cdot d\mathbf{r} = \frac{q\Phi}{\hbar c},$$

where  $\Phi$  is the magnetic flux through the closed contour  $\gamma$ . This shift of phase, due to non-zero magnetic potential, gives the **Aharonov-Bohm effect**.

Upon quantisation of radiation, one can write the **vector potential operator**:

$$\hat{\mathbf{A}} = C \sum_{\mathbf{k}, \lambda} \left( e^{i\mathbf{k} \cdot \mathbf{x}} \boldsymbol{\epsilon}_{\mathbf{k}, \lambda} \hat{a}_{\mathbf{k}, \lambda}(t) + e^{-i\mathbf{k} \cdot \mathbf{x}} \boldsymbol{\epsilon}_{\mathbf{k}, \lambda}^* \hat{a}_{\mathbf{k}, \lambda}^\dagger(t) \right),$$

where the destruction/creation operators conform to the commutation relation

$$\left[ \hat{a}_{\mathbf{k}, \lambda}, \hat{a}_{\mathbf{k}', \lambda'}^\dagger \right] = \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\lambda, \lambda'}.$$

This relies upon the **Coulomb gauge**  $\nabla \cdot \mathbf{A} = 0$ . This gives that the EM field is a sum over modes of quantum harmonic oscillators.